Limits at Infinity; Horizontal Asymptotes

Definition: Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the value of $f(x)$ can be made arbitrarily close to L when x becomes sufficiently large.

Definition: Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to -\infty} f(x) = L$$

means that the value of $f(x)$ can be made arbitrarily close to L when x becomes sufficiently large negative.

Ex. $\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$

Ex. Neither $\lim_{x\to\infty} \sin x$ nor $\lim_{x\to\infty} \cos x$ exists.

Definition: For a real number* L, the line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to -\infty} f(x) = L$$

* That is, L is a finite number; recall that ∞ or $-\infty$ are not real numbers.

> *Note*: Therefore, the graph of a function can have at most 2 horizontal asymptotes.

Ex. The line $y = 0$ is the horizontal asymptote of $f(x) = \frac{1}{x}$

Theorem: If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} x^r = \infty$$

Ex. $\lim_{x\to\infty} \sqrt[10]{x} = \lim_{x\to\infty} x^{1/10} = \infty$

Theorem: If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Ex. $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = \lim_{x\to\infty} \frac{1}{x^{1/2}} = 0$

Ex. $\lim_{x\to-\infty} \frac{5}{\sqrt[3]{x}} = 5 \lim_{x\to-\infty} \frac{1}{x^{1/3}} = 5(0) = 0$

Limits at Infinity of Rational Functions:

According to the above theorem, if n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 = \lim_{x\to-\infty} \frac{1}{x^n}$$

This fact can be used to find the limits at infinity for any rational function.

Ex. $\lim_{x\to\infty} \frac{x^2+100x+1000}{x^3-10} = 0$

The limit as $x \to -\infty$ is the same. The graph of the function in the limit above has a horizontal asymptote $y = 0$.

Ex. $\lim_{x\to\infty} \frac{-5x^4+8x^2-7}{3x^4+10x^3-2x} = \frac{-5}{3}$

Therefore, the graph of the function in the limit above has a horizontal asymptote $y = -5/3$. (What is the limits as $x \to -\infty$?)

Ex. $\lim_{x\to\infty} \frac{x^3-4x^2+4x-5}{2x^2-1} = \infty$ and $\lim_{x\to -\infty} \frac{x^3-4x^2+4x-5}{2x^2-1} = -\infty$