Probability Models
General Probability Rules

- Coin tossing;
- Probability models
  - Sample spaces and events
  - Venn diagrams
  - Basic probability rules
  - Assigning probabilities: a finite sample space
  - Assigning probabilities: intervals of outcomes
  - Independence and the multiplication rule

Example

Coin tossing.

In ther early 1700's, French naturalist George Louis Leclerc Buffon observed 2,048 heads in 4,040 tosses.

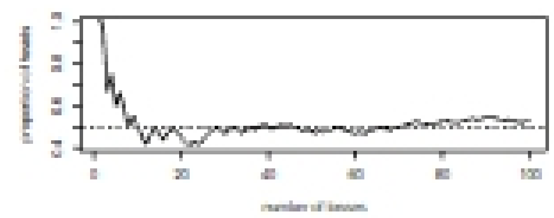
Relative frequency $= \frac{2048}{4040} = 0.5069$.

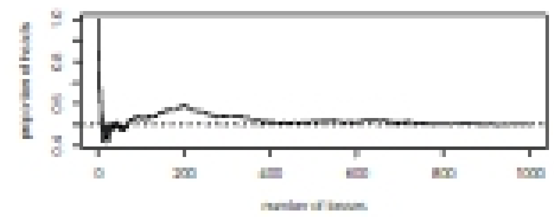
In the late 1800's, English statistician Karl Pearson observed 12,012 heads in 24,000 tosses.

Relative frequency $= \frac{12012}{24000} = 0.5005$.

While in a prison camp in Denmark during World War I, English mathematician John Kerrich observed 5,067 heads in 10,000 tosses.

Relative frequency $= \frac{5067}{10000} = 0.5067$.





Randomness and Probability

Randomness $\neq$ Complete Chaos!

A phenomenon is said to be **random** if individual outcomes are uncertain but there is a regular distribution of outcomes in a large number of repetitions.

The **probability** of any outcome of a random phenomenon is the proportion of times the outcome would occur in a very long series of repetitions.

Probability Models

The **sample space**, denoted by S, is the set of all possible outcomes of a random phenomenon.

- Toss a coin and record the side facing up. Then $S = \{\text{Head}, \text{Tail}\} = \{H, T\}$.
- Toss a coin twice. Record the side facing up each time. Then $S = ?$
- Toss a coin twice. Record the number of heads in the two tosses. Then $S = ?$

An **event** is an outcome or a set of outcomes of a random phenomenon (i.e. a subset of the sample space).

Toss a coin three times. Then
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

- Let A be the event that we got exactly two tails. Then $A = ?$
- Let B be the event that we got at least one head. Then $B = ?$

A **probability model** is a mathematical description of a random phenomenon consisting of two parts: a sample space S and a way of assigning probabilities to events.

The **probability** of an event A, denoted by $P(A)$, can be considered the long run relative frequency of the event A.

Set Notation

Suppose A and B are events in the sample space S. Then,

- $(A \cup B) \equiv (A \text{ or } B) \equiv$
  the set of all outcomes in A, or in B, or in both
- $(A \cap B) \equiv (A \text{ and } B) \equiv$
  the set of all outcomes that are in A AND in B
- $(A \cap B = \emptyset) \equiv A$ and B are disjoint $\equiv$
  A and B are mutually exclusive $\equiv$
  A and B have no outcomes in common
- $A^c \equiv$ the complement of $A \equiv$
  the event that A does not occur.

Example

Toss a coin twice. Let A be the event that we get 2 heads, B the event that we get exactly 1 tail, and C the event that we get at least one head. So,

$$A = \{HH\} \qquad B = \{TH, HT\} \qquad C = \{HH, HT, TT\}$$

- $A^c = ?$
- $B^c = ?$
- $A \cap B = ?$
- $A \cup B = ?$
- $A \cap C = ?$
- $B \cup C = ?$

Rules of Probability

1. For any event A, $0 \leq P(A) \leq 1$.
2. $P(S) = 1$.
3. For any event A, $P(A^c) = 1 - P(A)$.
4. If $P(A \cap B) = \emptyset$, then

$$P(A \cup B) = P(A) + P(B)$$

More generally,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probabilities in a Finite Sample Space

If the sample space is finite, each distinct event is assigned a probability. The probability of an event is the sum of the probabilities of the distinct outcomes making up the event.

If a random phenomenon has k equally likely outcomes, each individual outcome has probability $\frac{1}{k}$. For any event A,

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S}$$

Example

Roll a fair die and looking at the face value.

Sample space: $S = \{1, 2, 3, 4, 5, 6\}$
This is a finite sample space, and each outcome is equally likely. That is,

$$P(X = j) = 1/6, \ \forall j \in S$$

where X is the face value of the die after rolling.

$P(X \geq 5) = P(X = 5) + P(X = 6) = 1/6 + 1/6 = 1/3$
$P(X \leq 2) = ?$

Assigning Probabilities: Intervals of Outcomes

Example

A software random number generator is designed to produce a number between 0 and 1 chosen at random.

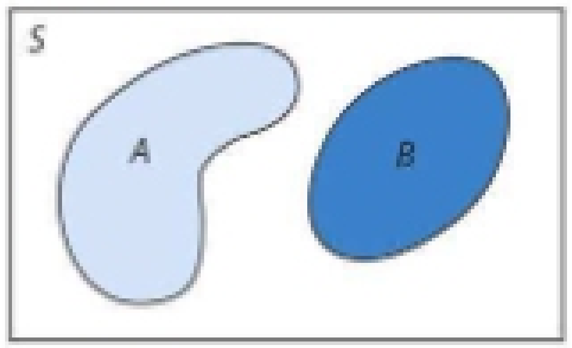
Sample space: $S = [0, 1]$

S is not finite, and the probability of an event, for example that the chosen number is between 0.3 and 0.7, is the area under the appropriate density curve. In this case, we would use the **uniform density curve**.

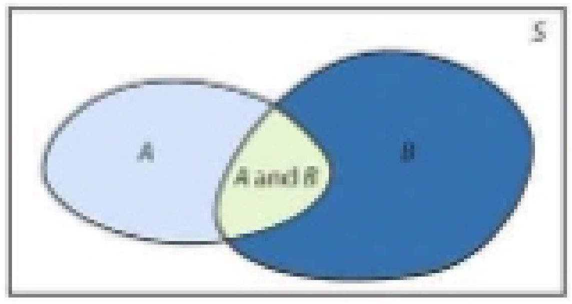
Recall the normal calculations from the first week. The normal density curve that we discussed is (along with a properly defined sample space) a probability model.

Venn Diagrams

A **Venn diagram** is a graphical representation of events in a sample space. The sample space S is represented as the rectangle and the events are areas within S. Below is a Venn diagram with two disjoint events.



Below is a Venn diagram with events A and B overlapping.



Independence and Disjointness

Two events A and B are **independent** if knowing that one occurs does not change the probability that the other occurs.

If A and B are independent, then

$$P(A \cap B) = P(A)\, P(B)$$

Events A and B are **disjoint** if they have no outcomes in common.

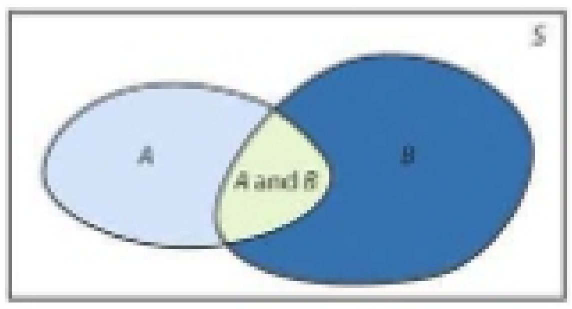
If A and B are disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

If A and B are disjoint, then the fact A occurs tells us that B can not occur. So disjoint events are not independent.

Independence can not be shown in a Venn diagram because it involves the probabilities of the events rather than just the outcomes that make up the events.

Conditional Probability: $P(B|A)$



Idea of $P(B|A)$: Given that A occurs, what is the probability that B also occurs?

$$P(B|A) \quad = \quad \frac{P(A \text{ and } B)}{P(A)}$$

Example

A deck of cards has 4 suits: $\heartsuit, \clubsuit, \spadesuit, \diamondsuit$. There are 13 cards in each suit: 2 through 10, jack, queen, king, and ace. So, there are $4 \times 13 = 52$ cards in the deck.