MA123, Chapter 8: Idea of the Integral (pp. 155-187, Gootman)

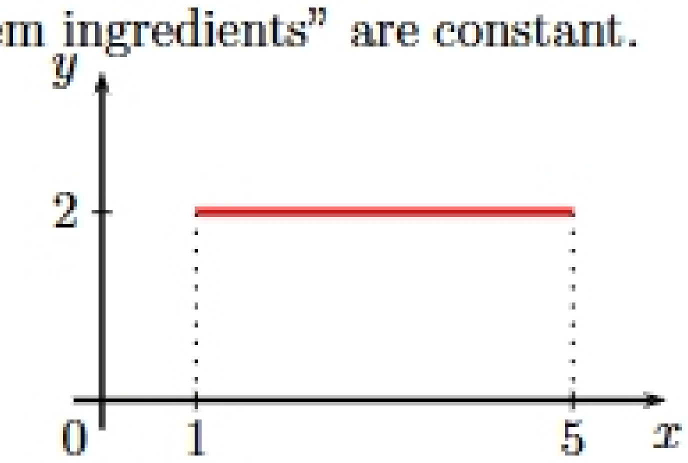
`Chapter Goals:`

- Understand the relationship between the area under a curve and the definite integral.
- Understand the relationship between velocity (speed), distance and the definite integral.
- Estimate the value of a definite integral.
- Understand the summation, or Σ, notation.
- Understand the formal definition of the definite integral.

`Assignments:` `Assignment 18` `Assignment 19`

▶ **The basic idea:** The first two problems are easy to solve as certain "problem ingredients" are constant.

Example 1 (Easy area problem): Find the area of the region in the xy-plane bounded above by the graph of the function $f(x) = 2$, below by the x-axis, on the left by the line $x = 1$, and on the right by the line $x = 5$.

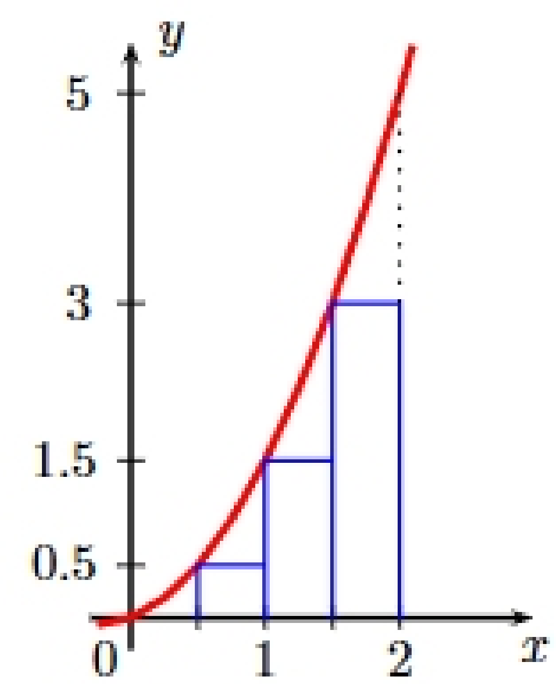


Example 2 (Easy distance traveled problem): Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

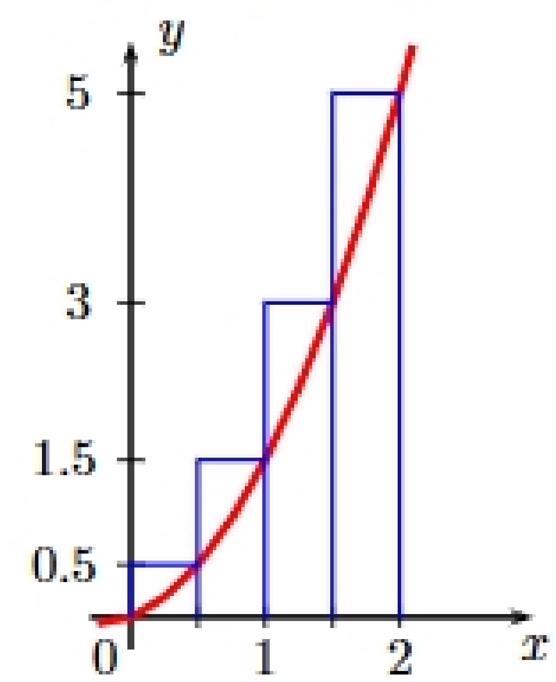
General philosophy: By means of the integral, problems similar to the previous ones can be solved when the ingredients of the problem are variable. In this Chapter, we learn how to *estimate* a solution to these more complex problems. *The key idea is to notice that the value of the function does not vary very much over a small interval, and so it is approximately constant over a small interval.* By the end of Chapter 9 we will be able to solve these problems *exactly*, and by the end of Chapter 10 we will be able to solve them both exactly and *easily*.

Example 3: Estimate the area under the graph of $y = x^2 + \frac{1}{2}x$ for x between 0 and 2 in two different ways:

(a) Subdivide the interval $[0, 2]$ into four equal subintervals and use the left endpoint of each subinterval as "sample point".

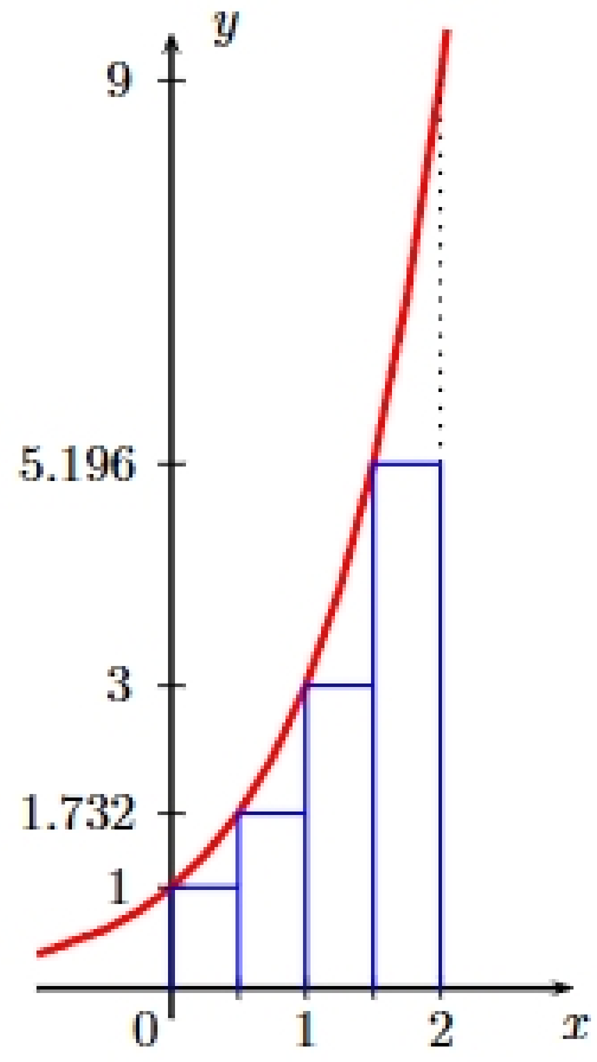


(b) Subdivide the interval $[0, 2]$ into four equal subintervals and use the right endpoint of each subinterval as "sample point".



Find the difference between the two estimates (right endpoint estimate minus left endpoint estimate).

Example 4: Estimate the area under the graph of $y = 3^x$ for x between 0 and 2. Use a partition that consists of four equal subintervals of $[0, 2]$ and use the left endpoint of each subinterval as a sample point.



Note: In the previous two examples we systematically chose the value of the function at one of the endpoints of each subinterval. However, since the guiding idea is that we are assuming that the values of the function over a small subinterval do not change by very much, then we could take the value of the function at <u>any</u> point of the subinterval as a good sample or representative value of the function. We could also have chosen small subintervals of different lengths. However, we are trying to establish a systematic procedure that works well in general.

Getting better estimates:

We can only expect the previous answers to be approximations of the correct answers. This is because the values of the function do change on each subinterval, even though they do not change by much.

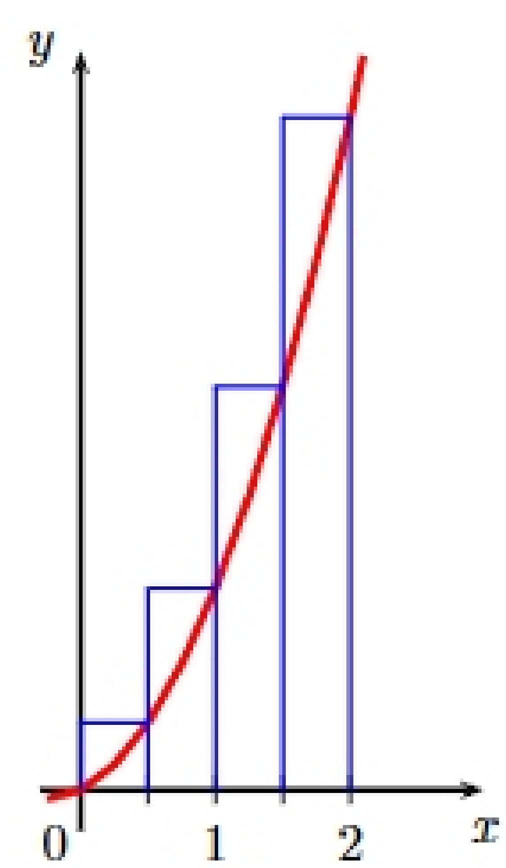
If, however, we replace the subintervals we used by "smaller" subintervals we can reasonably expect the values of the function to vary by much less on each thinner subinterval. Thus, we can reasonably expect that the area of each thinner vertical strip under the graph of the function to be more accurately approximated by the area of these thinner rectangles. Then if we add up the areas of all these thinner rectangles, we should get a much more accurate estimate for the area of the original region.

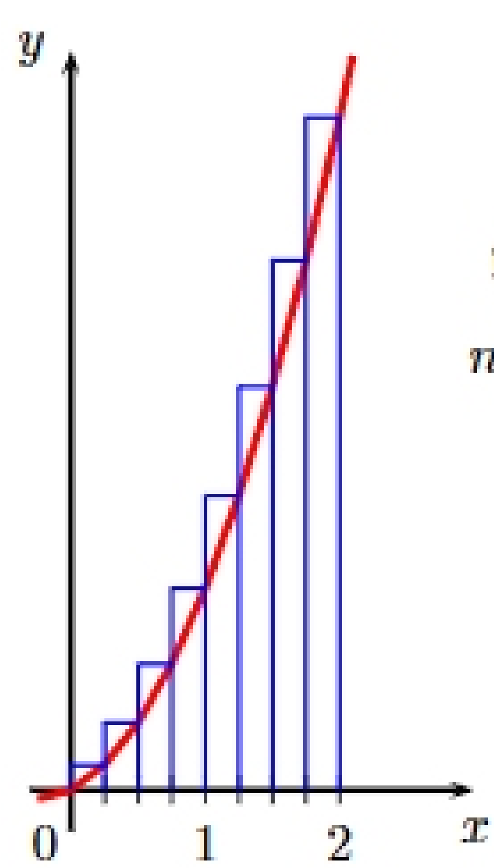
Here is Example 3(b), revisited:

$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 4$ equal subintervals

Area ≈ 5





$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 8$ equal subintervals

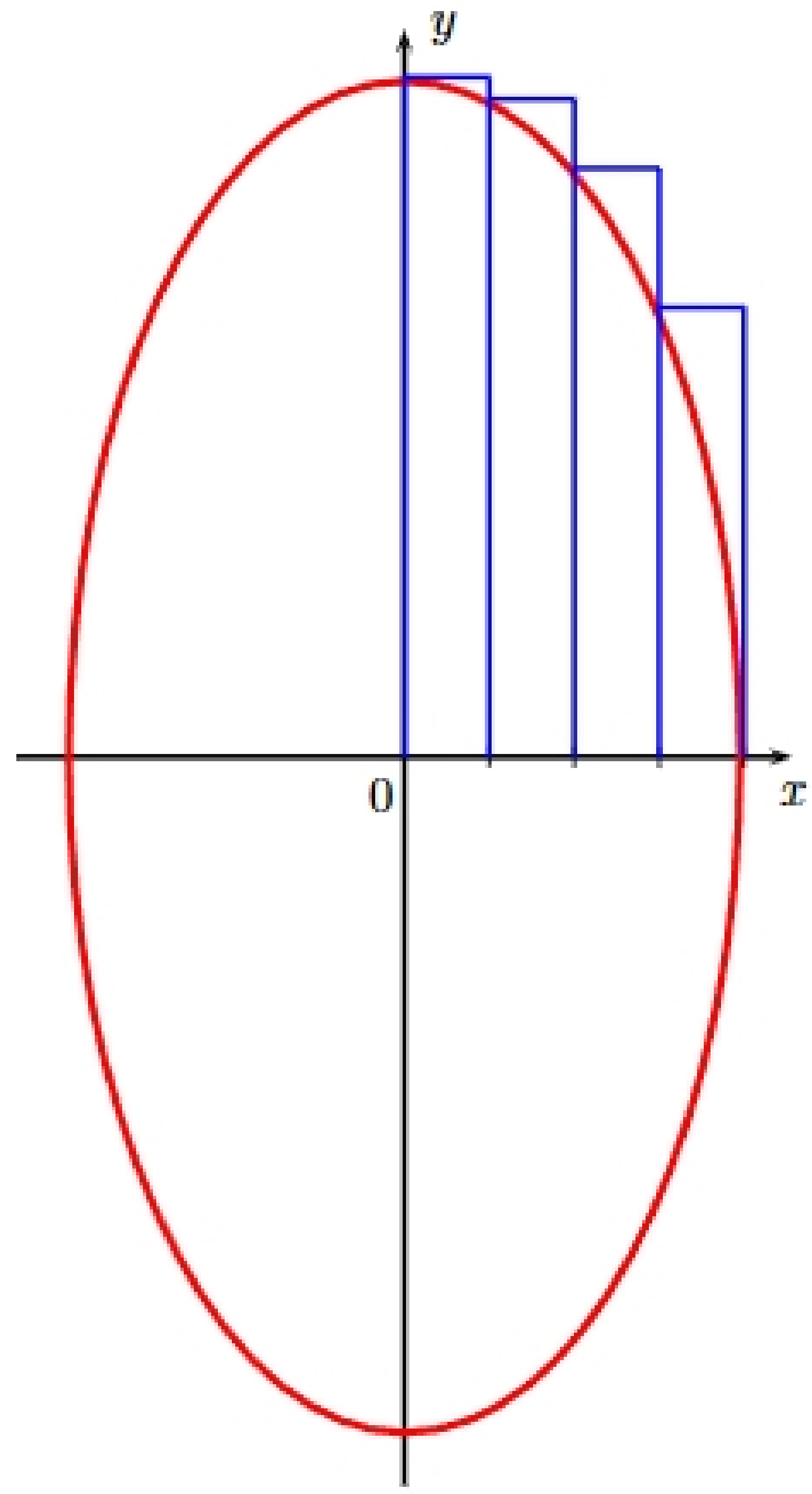
Area ≈ 4.3125

We will see later that the exact value of the area under consideration in Example 3 is $\frac{11}{3} \approx 3.66$.

Example 5: Estimate the area of the ellipse given by the equation

$$4x^2 + y^2 = 49$$

as follows: The area of the ellipse is 4 times the area of the part of the ellipse in the first quadrant (x and y positive). Estimate the area of the ellipse in the first quadrant by solving for y in terms of x. Estimate the area under the graph of y by dividing the interval $[0, 3.5]$ into four equal subintervals and using the left endpoint of each subinterval.

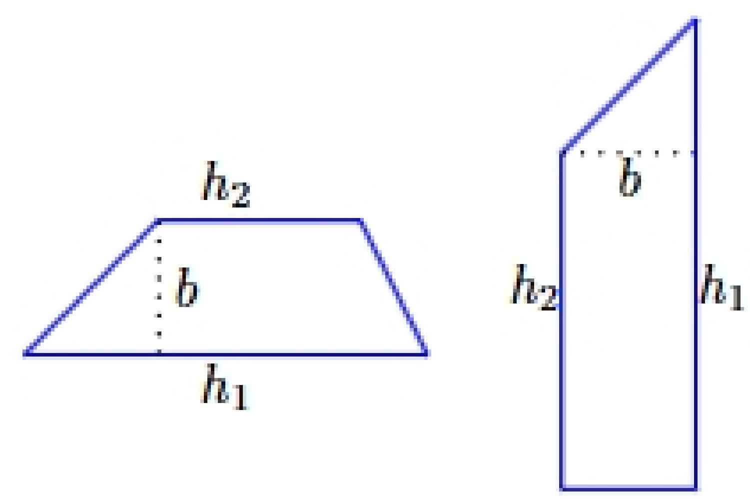


The area of the ellipse (using the above method) is approximately ________

Trapezoids versus rectangles:

We could use trapezoids instead of rectangles to obtain better estimates, even though the calculations get a little bit more complicated. This will occur in some of the latter examples. We recall that the area of a trapezoid is

$$\text{Area of a trapezoid} = \frac{(h_1 + h_2) \cdot b}{2}.$$



Example 6: A train travels in a straight westward direction along a track. The velocity of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are

time	0	0.1	0.2	0.3	0.4	0.5
velocity	0	10	15	18	20	25

We will see later that the total distance traveled by the train is equal to the area underneath the graph of the velocity function and lying above the t-axis. Compute the total distance traveled by the train during the first half hour by assuming the velocity is a linear function of t on the subintervals. (The velocity in the table is given in miles per hour.)