MA 111 Worksheet 3.2
Interest

1. Suppose we have \$1000, and we want to calculate its value after a 5% increase. There are two (equivalent) ways we have learned to do this:

$$\begin{aligned} \text{Way \#1:} \quad & \$1000 + \frac{5}{100} \cdot \$1000 \\ \text{Way \#2:} \quad & \$1000 \cdot 1.05 \end{aligned}$$

   We will be using the second way because it is simpler ...just a single multiplication.

   Note the specific type of question we usually are interested in: "How much money will I have after some period of time?" We are not only interested in the increase, but rather what the *total* amount will be.

2. **Annual percentage rate**
   This is the interest rate being charged/paid on a sum of money during one year's time. It is often abbreviated "APR."

   For the following examples, I will use an APR of $R = 5\%$ which, as a decimal, equals $r = \frac{5}{100} = 0.05$. *I will use R when expressing the rate as a percent, and r when expressing the rate as a decimal, where $r = \frac{R}{100}$.*

3. **Principal**
   The principal is the amount of money that is earning interest.

   For the following examples, I will use a starting principal of \$1000.

4. **Simple interest over one year**
   Example. $1.05 \cdot \$1000 = \$1050$. After one year of simple interest, we have \$1050.

5. **Simple interest over three years**
   For "simple interest," we just use the APR and the starting principal to figure interest earned.

   Example. Interest from first year: $0.05 \cdot \$1000 = \$50$.
   Interest from second year: $0.05 \cdot \$1000 = \$50$.
   Interest from third year: $0.05 \cdot \$1000 = \$50$.
   Total interest: \$150.
   Final amount: \$1150, which equals $\$1000 + (3 \cdot 0.05 \cdot \$1000)$ or $\$1000(1 + 3 \cdot 0.05)$.

   Formula. If principal P is invested at simple annual interest rate $R\%$ per year for t years, the final amount A is given by:

$$A = P(1 + tr),$$

   where $r = \frac{R}{100}$.

6. **Compound interest over three years, compounding annually**
For "compound interest," the interest that has already been earned is added to the principal. That is, we will earn interest on the interest we've already accrued.

Example. Interest from first year: $0.05 \cdot \$1000 = \$50$.
Principal after first year: \$1050.
Interest from second year: $0.05 \cdot \$1050 = \$52.50$.
Principal after second year: \$1102.50.
Interest from third year: $0.05 \cdot \$1102.50 = \$55.13$.
Principal after third year: \$1157.63.

Shorter: We can combine the two steps (interest and revised principal) into one step:
After the first year: $\$1000 \cdot 1.05 = \$1050$.
After the second year: $\$1050 \cdot 1.05 = \$1102.50$.
After the third year: $\$1102.50 \cdot 1.05 = \$1157.63$.

Even shorter: We can combine these three steps into one:
After three years: $\$1000 \cdot 1.05 \cdot 1.05 \cdot 1.05 = \$1157.63$.

Shortest!: We can use an exponent to indicate the number of years (that is, the number of factors of 1.05):
After three years: $\$1000 \cdot (1.05)^3 = \$1157.63$.

Formula. If principal P is invested at annual interest rate $R\%$ compounded every year for t years, the final amount A is given by:

$$A = P(1+r)^t,$$

where $r = \frac{R}{100}$.

7. **Compound interest over twenty years, compounding annually**
Example. $\$1000 \cdot (1.05)^{20} = \$2653.30$.

8. **Compound interest over one year, compounding monthly**
We first have to find the monthly interest rate. To do this, divide the APR r by 12.

Example. After one month: $\$1000 \cdot \left(1 + \frac{0.05}{12}\right) = \$1004.17$.

After twelve months: $\$1000 \cdot \left(1 + \frac{0.05}{12}\right)^{12} = \$1051.16$.

9. **Compound interest over t years, compounding monthly**

Formula. If principal P is invested at annual interest rate $R\%$ compounded every month for t years, the final amount A is given by:

$$A = P\left(1 + \frac{r}{12}\right)^{12t},$$

where $r = \frac{R}{100}$.

Example. After three years: $\$1000 \cdot \left(1 + \frac{0.05}{12}\right)^{36} = \$1161.47$.

10. **Compound interest over t years, compounding daily**

    Formula. If principal P is invested at annual interest rate $R\%$ compounded every day for t years, the final amount A is given by:

    $$A = P\left(1 + \frac{r}{365}\right)^{365t},$$

    where $r = \frac{R}{100}$.

    Example. After three years: $\$1000 \cdot \left(1 + \frac{0.05}{365}\right)^{1095} = \$1161.82$.

11. **Compound interest over t years, compounding continuously**

    We might consider compounding more often. How much does this help us?

    Example. Compound hourly. After three years we have $\$1000(1+\frac{.05}{8760})^{3 \cdot 8760} = \$1161.833745$.

    Example. Compound every minute. After three years we have $\$1000(1+\frac{.05}{525600})^{3 \cdot 525600} = \$1161.834234$.

    It is possible to imagine, in the limit, compounding continuously, all the time. It turns out that a simple formula involving the number e results:

    Formula. If principal P is invested at annual interest rate $R\%$ compounded continuously for t years, the final amount A is given by:

    $$A = Pe^{rt},$$

    where $r = \frac{R}{100}$.

    On my calculator, the "e" button is below the "π" button.

    Example. After three years: $\$1000 \cdot e^{0.15} = \$1161.834243$.

12. Reviewing the above results, we see there is a big jump in the final amount when we went from simple to compound interest (compounding annually). There is another big jump when we compound monthly instead of annually.

    Beyond that, there is not a lot to be gained from using shorter time periods for compounding. In practice, most banks will use monthly compounding for everything (but may use daily compounding; e.g., for outstanding credit card balances).