1 Introduction

The behavior of physical systems is described by various fundamental laws of nature, such as Newton's laws of motion, yielding mathematical equations which *approximate* (or model) the behavior of the physical system. We subsequently analyze these mathematical models in hopes of describing the dynamical behavior of the physical system. The goal of vibration analysis, and mathematical analysis in general, is to provide a predictive model. That is, a mathematical model which can be used to accurately simulate, and even predict the dynamical behavior of a physical system. This model can subsequently be used to better understand the physical behavior and adjust the engineering design to satisfy performance criteria.

Vibration Analysis Procedure

Step 1: Mathematical Modeling. What do we include in our model? For example, considering an automobile on an uneven road, do we model the vehicle as a single, two, or a multi-degree-of-freedom system? How do we model the road? The suspension system? The tires?

Step 2: Derivation of Governing Equations. Given the above modeling assumptions, what physical laws do we use to obtain a mathematical model?

Step 3: Solution of the Governing Equations. Can we solve the mathematical model in closed form? approximately? numerically?

Step 4: Interpretation of Results. What information does the analysis of our model provide, and is it consistent we experimental results? Can we use this information to predict future behavior or design a better system?

1.1 Spring-mass System

We begin with the simplest oscillatory system—a mass suspended by a linearly elastic spring under the influence of gravity with no other external forces. We further assume that the motion is confined to a single direction, say the $\hat{\boldsymbol{\jmath}}$ direction. If $x(t)$ measures the displacement of the mass from the equilibrium position (when the spring is unstretched), the acceleration of the mass is:

$${}^{\mathcal{F}}\mathbf{a}_{P} = \ddot{x}\,\hat{\boldsymbol{\jmath}}.$$

Because the spring is linear, the force generated by the spring on the mass is determined by the constitutive equation:

$$\boldsymbol{F}_{P} = -kx\,\hat{\boldsymbol{\jmath}} - mg\hat{\boldsymbol{\jmath}},$$

where k is known as the spring constant and mg is the magnitude of the gravitational force. Using Newton's laws of motion, we determine that the motion of the mass is governed by the equation:

$$\begin{aligned} \sum \boldsymbol{F}_{P} &= m\,{}^{\mathcal{F}}\mathbf{a}_{P}, \\ -(kx + mg)\,\hat{\boldsymbol{\jmath}} &= m\ddot{x}\,\hat{\boldsymbol{\jmath}}, \end{aligned}$$

which can be written as:

$$m\ddot{x} + kx = mg. \tag{1}$$

The solution to this equation, a linear, constant coefficient, inhomogeneous ordinary differential equation, can be written as:

$$x(t) = x_H(t) + x_P(t),$$

where $x_P(t)$ and $x_H(t)$ are the particular and homogeneous solutions respectively. The particular solution is any one solution which satisfies the original differential equation while $x_H(t)$ satisfies the homogeneous equation:

$$m\ddot{x} + kx = 0.$$

We seek a homogeneous solution of the form $x_H(t) = c\ e^{\lambda t}$. Substitution into Eq. (1) yields:

$$\left(m\lambda^2 + k\right)\ c\ e^{\lambda t} = 0.$$

This equation has a solution only if the characteristic equation:

$$m\lambda^2 + k = 0,$$

holds, which is satisfied for:

$$\lambda = \pm i\sqrt{\frac{k}{m}}.$$

Therefore, the solution to this homogeneous equation is:

$$x_H(t) = c_1\ e^{i\sqrt{(k/m)}\ t} + c_2\ e^{-i\sqrt{(k/m)}\ t},$$

where c_1 and c_2 are arbitrary complex constants. For $x(t)$ to be real, c_1 and c_2 must be complex conjugates of one another, so that, with $c_1 = c$ and $c_2 = \bar{c}$, the solution takes the form:

$$\begin{aligned} x_H(t) &= c\ e^{i\sqrt{(k/m)}\ t} + \bar{c}\ e^{-i\sqrt{(k/m)}\ t}, \\ &= A\ \sin\left(\sqrt{\frac{k}{m}}\ t + \phi\right). \end{aligned}$$

Since the external forcing mg is independent of time, we recognize that:

$$x_P(t) = \frac{mg}{k},$$

solves the original model and therefore serves as a particular solution. Consequently, the total solution can be written as:

$$\begin{aligned} x(t) &= x_P(t) + x_H(t), \\ &= \frac{mg}{k} + A\ \sin\left(\sqrt{\frac{k}{m}}\ t + \phi\right), \end{aligned}$$

subject to initial conditions on the position and velocity. One such set of initial conditions is:

$$x(0) = 0, \qquad \dot{x}(0) = 0,$$

which physically imply that the mass is released from rest at the unstretched position of the spring. Substitution into the above solution yields:

$$\begin{aligned} 0 = x(0) &= \frac{mg}{k} + A\ \sin(\phi), \\ 0 = \dot{x}(0) &= A\sqrt{\frac{k}{m}}\ \cos(\phi). \end{aligned}$$

These two equations can be solved for the two unknown constants A and ϕ as:

$$A = \frac{mg}{k}, \qquad \phi = -\frac{\pi}{2},$$

and the response of the system, subject to these initial conditions, is described by:

$$\begin{aligned} x(t) &= \frac{mg}{k} + \frac{mg}{k}\ \sin\left(\sqrt{\frac{k}{m}}\ t - \frac{\pi}{2}\right), \\ &= \frac{mg}{k}\left[1 - \cos\left(\sqrt{\frac{k}{m}}\ t\right)\right]. \end{aligned}$$

1.2 A Description of Harmonic Motion

Engineering vibration is primarily concerned with the periodic, oscillatory response of mechanical systems. The dynamical behavior of simple oscillatory systems can be described by the sine and cosine functions, that is, harmonic functions of the form:

$$\begin{aligned} x(t) &= A\ \sin(\omega t + \phi), \\ &= a\ \sin(\omega t) + b\ \cos(\omega t), \\ &= c\ e^{i\omega t} + \bar{c}\ e^{-i\omega t}, \end{aligned}$$

where (a, b) and (A, ϕ) are real constants and c is complex, with $\bar{c}$ its complex conjugate. These three representations are, in fact, equivalent, with:

$$a = A\ \cos\phi, \qquad b = A\ \sin\phi,$$

and:

$$c = \frac{A}{2}\ e^{i(\phi - \frac{\pi}{2})} = \frac{A}{2}\ (\sin\phi - i\cos\phi).$$

In the above function, ω is known as the frequency of vibration, measured in radians per unit time. A is the vibrational amplitude, while ϕ, the phase angle, shifts the response in time. The latter representation arises from Euler's identities:

$$\cos\tau = \frac{e^{i\tau} + e^{-i\tau}}{2}, \qquad \sin\tau = \frac{e^{i\tau} - e^{-i\tau}}{2i}.$$

The period of oscillation, or the time required for the response to repeat itself, is defined as T and is found to be:

$$T = \frac{2\pi}{\omega}.$$