$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt}$$

Section 16.3 The **Fundamental Theorem for line integrals.**

Theorem. Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

Conservative vector fields. A vector field $\vec{F}$ is conservative, if there is a function u such that $\boxed{\nabla u = \vec{F}}$

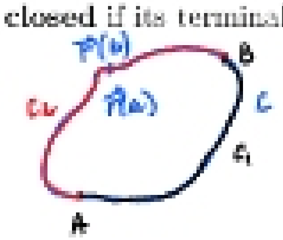
Independence of path.

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and the terminal point B.

In general, $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \ne \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$. But, according the the Theorem, if ∇f is continuous, then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$. In other words, **the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.**

In general, if $\mathbf{F}$ is a continuous vector-field with domain D, we say that the line integral is **independent of path** if $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points. **Line integrals of conservative vector fields are independent of path.**

A curve is called **closed** if its terminal point coincides with its initial point, that is $\mathbf{r}(a) = \mathbf{r}(b)$.



$C: \vec{r}(t), \quad a \le t \le b$

If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A.

Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \underbrace{\int_{C_1} \mathbf{F}\cdot d\mathbf{r}}_{f(B)-f(A)} + \underbrace{\int_{C_2} \mathbf{F}\cdot d\mathbf{r}}_{f(A)-f(B)} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{-C_2} \mathbf{F}\cdot d\mathbf{r} = 0$$

where $\vec{F} = \nabla f$

if $\vec{F}$ is a conservative vector field. Then
$\oint_C \vec{F}\cdot d\vec{r} = 0$, and $\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ [here $\nabla f = \vec{F}$]
$\boxed{f(x,y,z)}$ is a potential function for $\vec{F}$

Also we can show that if $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D.

> **Theorem.** $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path in D.

Now we assume that D is **open** (for every point P in D there is a disk with center P that lies entirely in D) and **connected** (any two points in D can be joined by a path that lies in D).

> **Theorem.** Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Definition. A curve is **simple** if it does not cross itself anywhere between its endpoints.
Definition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D (simply-connected region contains no hole and cannot consist of two separate pieces).

> **Theorem.** Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and
> $$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \quad , \quad \vec{F} = \langle P, Q\rangle$$
> Then $\mathbf{F}$ is conservative.

Example 1.

1. If $\mathbf{F} = <\underbrace{2xy^3}_{P}, \underbrace{3x^2y^2}_{Q}>$, find a function f such that $\nabla f = \mathbf{F}$.

$P(x,y) = 2xy^3$, $\quad Q(x,y) = 3x^2y^2$

$\frac{\partial P}{\partial y} = 6xy^2$, $\frac{\partial Q}{\partial x} = 6xy^2$ match!

$\vec{F}$ is conservative.

There is a function $f(x,y)$ such that $\nabla f(x,y) = \vec{F}$

$< f_x, f_y > = < 2xy^3, 3x^2y^2 >$

$\int f_x dx = \int 2xy^3 dx \Rightarrow f(x,y) = \boxed{x^2y^3} + g(y)$

$\int f_y dy = \int 3x^2y^2 dy \Rightarrow f(x,y) = \boxed{x^2y^3} + h(x)$, $g(y) = h(x) = C$ (set $C=0$)

$\boxed{f(x,y) = x^2y^3}$

2. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C given by $\mathbf{r}(t) = <\sin t, t^2+1>$, $0 \le t \le \pi/2$.

$\vec{r}(0) = <\sin 0, 1> = (0,1)$ | $\vec{r}(\frac{\pi}{2}) = (\sin\frac{\pi}{2}, \frac{\pi^2}{4}+1) = (1, \frac{\pi^2}{4}+1)$

Fundamental Theorem for line integrals
$\int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(1, \frac{\pi^2}{4}+1) - f(0,1) = \boxed{\left(\frac{\pi^2}{4}+1\right)^3}$

$f(x,y) = x^2y^3$

Example 2.

1. If $\mathbf{F} = \langle \underbrace{2xz + \sin y}_{P}, \underbrace{x\cos y}_{Q}, \overbrace{x^2}^{R} \rangle$, find a function f such that $\nabla f = \mathbf{F}$.

assume that there exists a function $f(x,y,z)$ such that

$\nabla f = \vec{F}$

$\langle f_x, f_y, f_z \rangle = \langle 2xz + \sin y, x\cos y, x^2 \rangle$

$\int f_x dx = \int (2xz + \sin y)\,dx \rightarrow f(x,y,z) = \boxed{x^2 z} + \boxed{x\sin y} + g(y,z)$

$\int f_y dy = \int (x\cos y)\,dy \rightarrow f(x,y,z) = \boxed{x\sin y} + h(x,z)$

$\int f_z dz = \int x^2\,dz \rightarrow f(x,y,z) = \boxed{x^2 z} + k(x,y)$

$$\boxed{f(x,y,z) = x^2 z + x\sin y}$$

2. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C given by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

$\vec{r}(0) = \langle \cos 0, \sin 0, 0 \rangle = (1,0,0)$

$\vec{r}(2\pi) = \langle \cos 2\pi, \sin 2\pi, 2\pi \rangle = (1,0,2\pi)$

$\vec{F}$ is conservative

By the Fundamental Theorem for line integrals

$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(2\pi)) - f(\vec{r}(0))$, where $f(x,y,z) = x^2 z + x\sin y$

$= f(1,0,2\pi) - f(1,0,0)$

$= \underbrace{1^2(2\pi) + 1\cdot\sin 0}_{f(1,0,2\pi)} \underbrace{- \quad 1^2(0) + 1\cdot \sin 0}_{-f(1,0,0)} = \boxed{2\pi}$

Example 3. Show that the line integral $\boxed{\int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)\,dy}$ is independent of path and evaluate the integral if C is any path from (-1,0) to (5,1).

$P(x,y) = 2x\sin y$, $Q(x,y) = x^2\cos y - 3y^2$

Check if $\vec{F} = \langle 2x\sin y, x^2\cos y - 3y^2 \rangle$ is conservative.

$\frac{\partial P}{\partial y} = 2x\cos y \underset{\text{match!}}{=} \frac{\partial Q}{\partial x} = 2x\cos y \rightarrow F$ is conservative $\Rightarrow \int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)dy$ is independent of path.

Find a potential function f for $\vec{F}$ ($\nabla f = \vec{F}$)

$\langle f_x, f_y \rangle = \langle 2x\sin y, x^2\cos y - 3y^2 \rangle$

$\int f_x dx = \int 2x\sin y\,dx \rightarrow f(x,y) = \boxed{x^2\sin y} + g(y)$

$\int f_y dy = \int (x^2\cos y - 3y^2)dy \rightarrow f(x,y) = \boxed{x^2\sin y}\boxed{-y^3} + h(x)$

$$\boxed{f(x,y) = x^2\sin y - y^3}$$

$\int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)\,dy = f(5,1) - f(-1,0) = 5^2\sin 1 - 1^3 - (-1)^2\sin 0 + 0^3$

$= \boxed{25\sin 1 - 1}$