

Lecture on Logic and Proofs

Building blocks of logic: propositions: a \sim is a declarative sentence that is either true or false, but not both.

Examples: These are propositions:

1. September 9, 2009 is Wednesday.
2. September 9, 2009 is Friday.
3. Today is Wednesday.
4. $2^{2^{n-2}} + 1$ is a prime.

These are not propositions.

1. Don't sleep in my class.
2. What time is it?
3. What a nice day!
4. $x + 1 = 2$.

Basic propositions: p, q, r .

From old to new: they are defined by the truth table.

1. Negation $\neg p$.
2. Conjunction p and q : $p \wedge q$.
3. Disjunction p or q : $p \vee q$. (inclusive or)
4. Implication $p \rightarrow q$: If p , then q .
5. Biconditional statement $p \leftrightarrow q$: p if and only if q .

One can write truth table for any proposition. The truth value of a proposition only depends on the value of its primitives, not the exact content.

Order of logical operations: Negation first, \wedge and \vee before implications. (...) always first.

Example 1. Truth table for $q \wedge (\neg r \rightarrow p)$.

Solution. T only then

p	q	r
F	T	T
T	T	F
T	T	T

Example 2. 1. Truth table for $p \rightarrow q$ and $\neg p \vee q$.

2. Truth table for $p \wedge \neg p$.

3. Truth table for $p \rightarrow (p \vee q)$

Theorem 1. $p \rightarrow q \equiv \neg p \vee q$

Tautology T_0 : a compound proposition that is always true no matter what the truth values of its primitives.

Contradiction F_0 : a compound proposition that is always false no matter what the truth values of its primitives.

Definition 1. Two propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.

(Easy) logical equivalences. (To prove, use truth table.)

1. Double negative law. $\neg(\neg p) \equiv p$.
2. Commutative laws. $p \vee q \equiv q \vee p$, and $p \wedge q \equiv q \wedge p$.
3. Associative laws. $(p \vee q) \vee r \equiv p \vee (q \vee r)$, and $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.
4. Idempotent laws. $p \vee p \equiv p$ and $p \wedge p \equiv p$.
5. Identity laws. $p \wedge T_0 \equiv p$, and $p \vee F_0 \equiv p$.
6. Domination laws. $p \vee T_0 \equiv T_0$ and $p \wedge F_0 \equiv F_0$.
7. Negation laws. $p \vee \neg p \equiv T_0$, and $p \wedge \neg p \equiv F_0$.

The less-obvious ones – Those involving two different operations.

1. De Morgan's laws. $\neg(p \wedge q) \equiv \neg p \vee \neg q$, and $\neg(p \vee q) \equiv \neg p \wedge \neg q$. (Check the first one by truth table)
2. Distributive laws. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, and $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
3. Absorption laws. $p \wedge (p \vee q) \equiv p$, and $p \vee (p \wedge q) \equiv p$.

Example 3. Negate and simplify the proposition $p \rightarrow q$

Solution. Negation: $\neg(p \rightarrow q)$.

Simplification: $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv \neg(\neg p) \wedge \neg q \equiv p \wedge \neg q$.

Example 4. Negate and simplify the proposition $(p \wedge q) \rightarrow r$.

Solution.

$$\begin{aligned}\neg((p \wedge q) \rightarrow r) &\equiv \neg(\neg(p \wedge q) \vee r) \\ &\equiv \neg(\neg(p \wedge q) \wedge \neg r) \\ &\equiv (p \wedge q) \wedge \neg r.\end{aligned}$$

Example: If n is an integer and $3n + 2$ is odd, then n is odd.

Negation is stated as: n is an integer and $3n + 2$ is odd, and n is not odd. (Note that it is not an implication!)

Implication $p \rightarrow q$. Then
converse: $q \rightarrow p$.

Inverse: $\neg p \rightarrow \neg q$

Contrapositive: $\neg q \rightarrow \neg p$.

Theorem 2. $p \rightarrow q \equiv \neg q \rightarrow \neg p$.

Prove by Theorem 1.

(Optional):

Theorem 3 (The Principle of duality). *If s and t are propositions containing no connectives other than \wedge , \vee and \neg . Then $s \equiv t$ iff $s^d \equiv t^d$, where s^d is obtained from s by exchanging \wedge with \vee , and T_0 with F_0 .*