Mechanics
Physics 151

Lecture 3

Lagrange's Equations

(Goldstein Chapter 1)

Hamilton's Principle

(Chapter 2)

What We Did Last Time

- Discussed multi-particle systems
  - Internal and external forces
    - Laws of action and reaction
- Introduced constraints
  - Generalized coordinates
- Introduced Lagrange's Equations
  - ... and didn't do the derivation

→ Let's pick it up and start from there

Today's Goals

- Derive Lagrange's Eqn from Newton's Eqn
  - Use D'Alembert's principle
  - There will be a few assumptions
    - Will make them clear as we go
- Introduce Hamilton's Principle
  - Equivalent to Lagrange's Equations
    - Which in turn is equivalent to Newton's Equations
  - Does not depend on coordinates by construction
  - Derivation in the next lecture

Lagrange's Equations

Recipe

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$L(q, \dot{q}, t) \equiv T - V$$

Kinetic energy

Potential energy

Lagrangian

- Express $L = T - V$ in terms of generalized coordinates $\{q_j\}$, their time-derivatives $\{\dot{q}_j\}$, and time t
  - The potential $V = V(q, t)$ must exist
  - i.e. all forces must be conservative

Virtual Displacement

- Consider a system with constraints
  - Ordinary coordinates $\mathbf{r}_i$ ($i = 1...N$)
  - Generalized coordinates q_j ($j = 1...n$)

$$\begin{cases} \mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, \ldots, q_n, t) \\ \mathbf{r}_2 = \mathbf{r}_2(q_1, q_2, \ldots, q_n, t) \\ \quad \vdots \\ \mathbf{r}_N = \mathbf{r}_N(q_1, q_2, \ldots, q_n, t) \end{cases}$$

- Imagine moving all the particles slightly $\mathbf{r}_i \to \mathbf{r}_i + \delta\mathbf{r}_i$ $\quad q_j \to q_j + \delta q_j$

Virtual displacement

  - Note that $\delta\mathbf{r}_i$ must satisfy the constraints

$$\delta\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

3N coordinates not independent

n coordinates independent

D'Alembert's Principle

- From Newton's Equation of Motion

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \quad \Longrightarrow \quad \mathbf{F}_i - \dot{\mathbf{p}}_i = 0$$

- Part of the force $\mathbf{F}_i$ must be due to constraints

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$$

"applied" force

"constraint" force

  - Applied force is "known" $\mathbf{F}_i^{(a)} = \mathbf{F}_i^{(a)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_N, t)$
- Constraint force $\mathbf{f}_i$ (usually) does no work
  - Movement is perpendicular to the force $\mathbf{f}_i \delta\mathbf{r}_i = 0$
  - Exception: friction
- Now multiply $\mathbf{F}_i^{(a)} + \mathbf{f}_i - \dot{\mathbf{p}}_i = 0$ by $\delta\mathbf{r}_i$ and sum over i