The Inverse of a Square Matrix

In this section we discuss a procedure for finding the inverse of a matrix and show how the inverse can be used to help us solve a system of linear equations.

Let A be a square matrix of size n. A square matrix A^{-1} of size n such that $AA^{-1} = A^{-1}A = I_n$ is called the inverse of A. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note: Not every square matrix has an inverse. A matrix with no inverse is called singular.

Finding the Inverse of a Matrix

Given the n x n matrix *A*:

1. Adjoin the n x n identity matrix *I* to obtain the augmented matrix

$(A \mid I)$

2. Use a sequence of row operations to reduce $(A \mid I)$ to the form $(I \mid B)$, if possible.

The matrix *B* is the inverse of *A*.

Example 1: Find the inverse:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] R_1 + R_2 \to R_2 = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right] \tfrac{1}{5}R_2 \to R_2$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] -2R_2 + R_1 \to R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Example 2: Find the inverse of a 3 x 3 matrix.(Use Gauss-Jordan)

$$\mathbf{C} = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 3 & -2 \\ -1 & 2 & 3 \end{pmatrix} \quad \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 2 & 3 & -2 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -5 & 0 & -2 & 1 & 0 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array}\right] R_2 + R_3 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 6 & 2 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} -4R_2 + R_1 \to R_1 \\ -6R_2 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 5 & -4 & -4 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & -10 & 7 & -6 & -5 \end{array}\right] R_3\left(-\tfrac{1}{10}\right) \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 5 & -4 & -4 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & -7/10 & 3/5 & 1/2 \end{array}\right]$$

$$\begin{array}{l} 9R_3 + R_1 \to R_1 \\ -2R_3 + R_2 \to R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{13}{10} & 7/5 & 1/2 \\ 0 & 1 & 0 & 2/5 & -1/5 & 0 \\ 0 & 0 & 1 & -2/10 & 3/5 & 1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -13/10 & 7/5 & 1/2 \\ 2/5 & -1/5 & 0 \\ -7/10 & 3/5 & 1/2 \end{bmatrix}$$

Example 3: Find the inverse.

$$\mathbf{B} = \begin{pmatrix} 4 & 2 & 2 \\ -1 & -3 & 4 \\ 3 & -1 & 6 \end{pmatrix} \quad \left[\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ -1 & -3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array}\right] R_1 - R_3 \to R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & -1 \\ -1 & -3 & 4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & -1 \\ \underbrace{0 \quad 0 \quad 0} & & & 1 & 1 & -1 \\ & & & & & \end{array}\right]$$

No inverse, rows of zeros
on left and numbers on the
right

Matrices That Have No Inverses

If there is a row to the left of the vertical line in the augmented matrix containing all zeros, then the matrix does not have an inverse. Example 3 has this problem and does not have an inverse.

Formula for the Inverse of a 2X2 Matrix

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. **Suppose** $D = ad - bc$ **is not equal to zero. Then.** $\mathbf{A}^{-1}$ **exists and is given by** $\mathbf{A}^{-1} = \frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 4: Find the inverse of the following matrices.

a. $\mathbf{A} = \begin{pmatrix} -5 & 10 \\ 2 & 7 \end{pmatrix}$

$d = -5(7) - 10(2) = -35 - 20 = -55$

$A^{-1} = -\frac{1}{55}\begin{bmatrix} 7 & -10 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{55} & \frac{10}{55} \\ \frac{2}{55} & \frac{5}{55} \end{bmatrix} = \begin{bmatrix} -7/55 & \frac{2}{11} \\ \frac{2}{55} & \frac{1}{11} \end{bmatrix}$

b. $\mathbf{B} = \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}$

$D = 8(2) - (-4)(-4)$
$= 16 - 16 = 0$

no inverse for matrix B

The use of inverses to solve systems of equations is advantageous when we are required to solve more than one system of equations AX = B, involving the same coefficient matrix, A, and different matrices of constants, B.

$AX = B$

$A^{-1}X = A^{-1}B$

$X = A^{-1}B$