MATH 285 E1/F1 GRADED HOMEWORK SET 1
DUE WEDNESDAY SEPTEMBER 10 IN LECTURE

IT WOULD BE SO SWEET if you followed these instructions: Please put each problem on a **separate sheet** of paper with your **name and section (E1 or F1)**. If a problem runs multiple pages, please staple all the pages for a single problem together. Think of each problem as a separate assignment. This may be annoying, but it will greatly streamline the grading process, resulting in faster feedback for you. **Thank you!**

Section and problem numbers refer to *Differential Equations & Boundary Value Problems*, Fourth Edition, by Edwards and Penney.

(1) Let $f(x)$ be the function defined piece-wise as

$$f(x) = \begin{cases} x & \text{if } x \leq 5 \\ 5 & \text{if } x > 5 \end{cases}$$

Find the solution of the initial value problem

$$\frac{dy}{dx} = f(x), \quad y(0) = 100.$$

Hint: Your solution will also be defined piece-wise.

Solution:

$$y(x) = \begin{cases} 100 + \frac{1}{2}x^2 & \text{if } x \leq 5 \\ 100 - \frac{25}{2} + 5x & \text{if } x > 5 \end{cases}$$

To obtain this, first consider the range $x \leq 5$. The equation becomes $\frac{dy}{dx} = x$, with general solution $y(x) = \frac{1}{2}x^2 + C$. In order to satisfy the initial condition $y(0) = 100$, the constant must be $C = 100$. This gives the first part of the piece-wise definition.

Second, consider the range $x > 5$. The equation becomes $\frac{dy}{dx} = 5$, with general solution $y(x) = 5x + D$. We have to choose the value of D so that the two pieces match at $x = 5$ (that is, so that $y(x)$ is a continuous function). At $x = 5$, the formula $100 + \frac{1}{2}x^2$ has the value $100 + \frac{25}{2}$. So we need $5(5) + D = 100 + \frac{25}{2}$, hence $D = 100 - \frac{25}{2}$. This gives the second part of the piece-wise definition.

(2) Consider the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Sketch the slope field for this equation. What are the solution curves? *Hint*: You should recognize them as semi-familiar geometric shapes.

Solution: The slope field is perpendicular to the lines through the origin. The solution curves are the upper and lower half-circles centered at the origin. Note that an entire circle is not a solution curve because it does not define y as a function of x.

(3) Section 1.4, problem 22.
Solution:

$$y(x) = -3e^{x^4-x}.$$

Starting from $\frac{dy}{dx} = 4x^3y - y$, separate variables to obtain

$$\int \frac{dy}{y} = \int (4x^3 - 1)\, dx$$

$$\ln |y| = x^4 - x + C$$

$$y = \pm e^C e^{x^4-4} = De^{x^4-x}$$

(where the constant D absorbs the plus/minus sign and e^C). We now use the initial condition $y(1) = -3$.

$$-3 = y(1) = De^{1^4-1} = De^0 = D.$$

So $D = -3$.

(4) Section 1.5, problem 10 (Find the general solution valid for $x > 0$).
Solution:

$$y(x) = 3x^3 + Cx^{3/2}.$$

We first put the equation $2xy' - 3y = 9x^3$ into standard form

$$y' - \frac{3}{2x}y = \frac{9}{2}x^2$$

An integrating factor is

$$e^{\int -\frac{3}{2x}\, dx} = e^{-\frac{3}{2}\ln|x|} = (e^{\ln|x|})^{-3/2} = |x|^{-3/2}$$

Since we are restricting the domain to $x > 0$, we may simply use $x^{-3/2}$ as the integrating factor. Multiply through:

$$x^{-3/2}y' - \frac{3}{2}x^{-5/2}y = \frac{9}{2}x^{1/2},$$

$$\frac{d}{dx}(x^{-3/2}y) = \frac{9}{2}x^{1/2}.$$

Integrate:

$$x^{-3/2}y = 3x^{3/2} + C.$$

Solve for y:

$$y = 3x^3 + Cx^{3/2}.$$

(5) Section 1.6, problem 14.

Solution:

$$y(x) = \pm\sqrt{2Cx + C^2}.$$

Starting from $yy' + x = \sqrt{x^2 + y^2}$, use the substitution $u = x^2 + y^2$. Then $u' = 2x + 2yy'$, which we recognize as 2 times the left-hand side. The equation becomes

$$(1/2)u' = \sqrt{u}.$$

Separate variables:

$$\int (1/2)u^{-1/2}\,du = \int dx,$$
$$u^{1/2} = x + C,$$
$$u = (x + C)^2.$$

Finally, solve for y:

$$x^2 + y^2 = u = (x + C)^2,$$
$$y = \pm\sqrt{(x + C)^2 - x^2},$$
$$y = \pm\sqrt{2Cx + C^2}.$$