CMSC250, Spring 2004 — Homework 5 Answers

Due Wednesday, March 3 at the beginning of your discussion section.

You must write the solutions to the problems single-sided on your own lined paper, with all sheets stapled together, and with all answers written in sequential order or you will lose points.

Prove each of the following statements true or false. Remember, a counterexample may only be used to prove that a "for all" statement is false, and all counterexamples must include specific values and enough algebra/justification to show that they are truly counterexamples.

1. For all integers a, b, and c, if $a + b = c$ and $a \mid b$, then $a \mid c$.

   Answer: TRUE.
   Let a, b, and c be arbitrary integers.
   Assume $a + b = c$ and $a \mid b$.
   Since $a \mid b$, $\exists k \in \mathbf{Z}\ ak = b$ by definition of divides.
   $ak + a = b + a$ by adding a to both sides.
   $a(k + 1) = c$ by algebra and substitution.
   $k + 1 \in \mathbf{Z}$ by closure of the integers under addition.
   $a \mid c$ by definition of divides.
   $\forall a, b, c \in \mathbf{Z}\ (a + b = c) \wedge (a \mid b) \rightarrow (a \mid c)$ by closing of conditional world and generalizing from the generic particular.

2. $\forall a, b, c \in \mathbf{Z}\ [(a \mid c) \wedge (b \mid c)] \rightarrow [(a \mid b) \vee (b \mid a)]$

   Answer: FALSE.
   Counterexample: Let $a = 2$, $b = 3$, and $c = 6$.
   Then $a \mid c$ and $b \mid c$ since $2 \mid 6$ and $3 \mid 6$. However, $2 \nmid 3$ and $3 \nmid 2$.

3. $\forall a, b, c \in \mathbf{Z}\ [(a \mid b) \wedge (a \mid c)] \rightarrow [(b \mid c) \vee (c \mid b)]$

   Answer: FALSE.
   Counterexample: Let $a = 2$, $b = 4$, and $c = 6$.
   Then $a \mid b$ and $a \mid c$ since $2 \mid 4$ and $2 \mid 6$. However, $4 \nmid 6$ and $6 \nmid 4$.

4. If $a \mid b$ and $b \mid c$, then $a \mid c$, for any integers a, b, and c.

   Answer: TRUE.
   Let a, b, and c be arbitrary integers.
   Assume $a \mid b$ and $b \mid c$.
   Then $\exists m, n \in Z\ am = b \wedge bn = c$.
   Since $am = b$, then $(am)n = c$, which means $a(mn) = c$ by substitution and associativity.
   Since $mn \in \mathbf{Z}$ by closure of the integers under multiplication,
   $a \mid c$ by definition of divides.
   $\forall a, b, c \in \mathbf{Z}\ (a \mid b) \wedge (b \mid c) \rightarrow (a \mid c)$ by closing of conditional world and generalizing from the generic particular.

5. If x is an odd integer, then $x^2 - 1$ is divisible by 4.

Answer: TRUE.
Let x be an arbitrary odd integer.
Since x is odd, $\exists k \in \mathbf{Z}\ x = 2k+1$.
Then $x^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k$ by substitution and algebra.
Then $x^2 - 1 = 4(k^2 + k)$ by algebra.
Since $k^2 + k \in \mathbf{Z}$ by closure of the integers under positive exponentiation and addition, $4 \mid (x^2 - 1)$ by definition of divides.
$\forall x \in \mathbf{Z}^{\text{odd}}\ 4 \mid (x^2 - 1)$ by generalizing from the generic particular.

6. If a prime greater than 2 can be represented as $3k+2$ for some integer k, then that prime can be represented as $6m+5$ for some integer m.

Answer: TRUE.
Let p be an arbitrary prime greater than 2.
Assume $\exists k \in \mathbf{Z}\ p = 3k+2$.
Since k is an integer, k is either even or odd.

Assume k is even.
Then $\exists g \in \mathbf{Z}\ k = 2g$.
Then $p = 3k + 2 = 3(2g) + 2 = 6g + 2 = 2(3g+1)$ by substitution and algebra.
Since $3g + 1 \in \mathbf{Z}$ by closure of the integers under multiplication and addition, $2 \mid p$ by definition of divides.
However, this is a contradiction since p is a prime greater than 2.

So we know k is not even, and therefore, k is odd.
Then $\exists g \in \mathbf{Z}\ k = 2g+1$. Then $p = 3k+2 = 3(2g+1)+2 = 6g+3+2 = 6g+5$ by substitution and algebra.
$\forall p \in \mathbf{Z}^{\text{prime}>2}\ (\exists k \in \mathbf{Z}\ p = 3k+2) \rightarrow (\exists m \in \mathbf{Z}\ p = 6m+5)$ by closing conditional world and generalizing from the generic particular.

7. For any integer n, $n^3 \not\equiv_4 2$.

Answer: TRUE.
Let n be an arbitrary integer.
Assume (by the way of contradiction) that $n^3 \equiv_4 2$.
Therefore $4 \mid (n^3 - 2)$ by definition of mod, which means $\exists k \in \mathbf{Z}\ n^3 = 4k+2$.
By the quotient-remainder theorem, there exist unique integers q and r such that $n = 4q + r$ and $0 \leq r < 4$.

Case 1: assume $r = 0$.
So $n = 4q$ by substitution.
Then $n^3 = (4q)^3 = 64q^3$ by substitution and algebra.
Since $n^3 = 4k+2$, then we know $4k + 2 = 64q^3$ by substitution.
Then $1/2 = 16q^3 - k$ by algebra.
We know that $16q^3 - k \in \mathbf{Z}$ by closure of the integers under multiplication, addition, and positive exponentiation, however, $1/2 \notin \mathbf{Z}$. Contradiction.

Case 2: assume $r = 1$.
So $n = 4q + 1$ by substitution.
Then $n^3 = (4q+1)^3 = 64q^3 + 48q^2 + 12q + 1$ by substitution and algebra.
Since $n^3 = 4k+2$, then we know $4k + 2 = 64q^3 + 48q^2 + 12q + 1$ by substitution.

Then $1/4 = 16q^3 + 12q^2 + 3q - k$ by algebra.
We know that $16q^3 + 12q^2 + 3q - k \in \mathbf{Z}$ by closure of the integers under multiplication, addition, and positive exponentiation, however, $1/4 \notin \mathbf{Z}$. Contradiction.

Case 3: assume $r = 2$.
So $n = 4q + 2$ by substitution.
Then $n^3 = (4q + 2)^3 = 64q^3 + 96q^2 + 48q + 8$ by substitution and algebra.
Since $n^3 = 4k + 2$, then we know $4k + 2 = 64q^3 + 96q^2 + 48q + 8$ by substitution.
Then $1/4 = 16q^3 + 24q^2 + 12q + 2 - k$ by algebra.
We know that $16q^3 + 24q^2 + 12q + 2 - k \in \mathbf{Z}$ by closure of the integers under multiplication, addition, and positive exponentiation, however, $1/2 \notin \mathbf{Z}$. Contradiction.

Case 3: assume $r = 3$.
So $n = 4q + 3$ by substitution.
Then $n^3 = (4q + 3)^3 = 64q^3 + 144q^2 + 108q + 27$ by substitution and algebra.
Since $n^3 = 4k + 2$, then we know $4k + 2 = 64q^3 + 144q^2 + 108q + 27$ by substitution.
Then $-25/4 = 16q^3 + 36q^2 + 27q - k$ by algebra.
We know that $16q^3 + 36q^2 + 27q - k \in \mathbf{Z}$ by closure of the integers under multiplication, addition, and positive exponentiation, however, $-25/4 \notin \mathbf{Z}$. Contradiction.

So all four cases lead to contradictions. However, one of the cases must be true by the quotient-remainder theorem, which is in itself another contradiction, which means our original assumption must be false.
So $n^3 \not\equiv_4 2$.
$\forall n \in \mathbf{Z}\ n^3 \not\equiv_4 2$ by generalizing from the generic particular.

8. $\forall x \in \mathbf{Z}^{\text{even}}\ (3 \nmid x) \rightarrow (4 \mid x^2)$

Answer: Let x be an arbitrary even integer.
Assume that $3 \nmid x$.
By the quotient-remainder theorem, there exist unique integers q and r such that $x = 3q + r$ and $0 \leq r < 3$.
So $(x = 3q) \vee (x = 3q + 1) \vee (x = 3q + 2)$ by enumerating all the possibilities.

Case 1: assume $x = 3q$.
Then $3 \mid x$ by definition of divides, which is a contradiction. So case 1 can never occur.

Case 2: assume $x = 3q + 1$.
Since q is an integer, q is either odd or even.

Assume that q is even.
Then $\exists k \in \mathbf{Z}\ q = 2k$ by definition of even.
Then $x = 3(2k) + 1 = 6k + 1$ by algebra and substitution.
However, we claimed that x was even, which means $\exists m \in \mathbf{Z}\ x = 2m$. Then $6k + 1 = 2m$ by substitution.
$1/2 = m - 3k$ by algebra, which is a contradiction since $m - 3k \in \mathbf{Z}$ by closure of the integers under multiplication and subtraction, but $1/2 \notin \mathbf{Z}$.

So q cannot be even, and therefore q is odd (by disjunctive syllogism).
Then $\exists k \in \mathbf{Z}\ q = 2k + 1$ by definition of odd.
Then $x = 3(2k + 1) + 1 = 6k + 3 + 1 = 6k + 4$ by algebra and substitution.
$x^2 = (6k + 4)^2 = 36k^2 + 48k + 4 = 4(9k^2 + 12k + 1)$ by substitution and algebra.