Note Title

Problem 5

1(a). Fro conservation of energy $p_e = p_m$.
Substitute $p_e = v_b i = k_e \omega i$ and
$p_m = \tau^e \omega = k_t i \omega$ to obtain

$$\overbrace{k_e \cancel{\omega} \cancel{i}}^{p_e} = \overbrace{k_t \cancel{i} \cancel{\omega}}^{p_m}$$

It follows that $k_t = k_e$.

1(b) $$k_e = 0.804 \frac{mV}{RPM} \left(\frac{1V}{1000mV}\right)\left(\frac{1 rev}{2\pi \, rad}\right)\left(\frac{60 s}{1 min}\right)$$

$$\underline{k_e = 7.68 \times 10^{-3}} \frac{V}{rad/sec}$$

$$k_t = 1.088 \frac{oz \, in}{A} \left(\frac{0.254 m}{1 in}\right)\left(\frac{0.278 N}{1 oz}\right)$$

$$\underline{k_t = 7.68 \times 10^{-3}} \frac{N \cdot m}{A}$$

Using the fact that radians are unitless and substituting $1V = 1 \frac{N \cdot m}{C}$, $1A = \frac{1C}{s}$ yields

$$\frac{V}{rad/sec} = \frac{N \cdot m}{C/s} = \frac{N \cdot m}{A}.$$

It follows that

$$\boxed{k_e = k_t = 7.68 \times 10^{-3} \frac{V}{rad/sec}}$$

<u>Problem 5</u>

2. From lecture 2, a state-space representation of the motor is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$x = \begin{pmatrix} i \\ \omega \end{pmatrix}, \quad u = \begin{pmatrix} V_a \\ T_L \end{pmatrix}, \quad y = \begin{pmatrix} \omega \\ i \end{pmatrix}$$

$$A = \begin{pmatrix} -R/L & -K_e/L \\ K_t/J & -B/J \end{pmatrix} \qquad B = \begin{pmatrix} 1/L & 0 \\ 0 & -1/J \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The transfer function representation is

$$H(s) = C(sI - A)^{-1}B + D$$

To determine $H(s)$, first find $(sI-A)^{-1}$:

$$sI - A = \begin{pmatrix} s + R/L & K_e/L \\ -K_t/J & s + B/J \end{pmatrix}$$

$$\det(sI - A) = (s + R/L)(s + B/J) - (K_e/L)(-K_t/J)$$
$$= s^2 + \left(\frac{R}{L} + \frac{B}{J}\right)s + \frac{RB + K_e K_t}{LJ}$$

Problem 5.

2. $$(sI-A)^{-1} = \frac{1}{\det(sI-A)}\begin{pmatrix} s+B/J & -k_e/L \\ K_t/J & s+R/L \end{pmatrix}.$$

$$(sI-A)^{-1}B = (sI-A)^{-1}\begin{pmatrix} 1/L & 0 \\ 0 & -1/J \end{pmatrix}$$

$$= \frac{1}{\det(sI-A)}\begin{pmatrix} s/L+B/JL & k_e/JL \\ K_t/JL & -s/J-R/LJ \end{pmatrix}$$

$$C(sI-A)^{-1}B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(sI-A)^{-1}B$$

$$= \frac{1}{\det(sI-A)}\begin{pmatrix} K_t/JL & -s/J-R/LJ \\ s/L+B/JL & K_e/JL \end{pmatrix}$$

Using the expression derived for $\det(sI-A)$ and the fact $D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ yields

$$H(s) = C(sI-A)^{-1}B + \cancel{D}^{\,0}$$

$$\begin{pmatrix} \Omega/V_a & \Omega/T_L \\ I/V_a & I/T_L \end{pmatrix} = \frac{\begin{pmatrix} K_t/JL & -s/J-R/LJ \\ s/L+B/JL & K_e/JL \end{pmatrix}}{s^2+\left(\frac{R}{L}+\frac{B}{J}\right)s+\frac{RB+K_eK_t}{LJ}}$$