Note Title

Problem 11

1. Given:
   - State-space system: $\dot{x} = Fx$, $x(0)$
   - F has eigenvalues λ_1, λ_2 with associated eigenvectors v_1, v_2
   - From Problem Set 3 Problem 10, if $P = (v_1, v_2)$ then
$$P^{-1} F P = \tilde{F} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (\square)$$

From lecture 5, the zero-input solution is
$$x(t) = e^{Ft} x(0) \quad \text{for } t \geq 0 \quad (\Delta)$$

Use equation $(\square)$ to express F in terms of the diagonal matrix $\tilde{F}$:
$$P(P^{-1} F P) P^{-1} = P(\tilde{F}) P^{-1}$$
$$F = P \tilde{F} P^{-1}$$

Replace F in equation (Δ) with $\tilde{F}$
$$\boxed{x(t) = e^{P\tilde{F}P^{-1} t} x(0) \quad t \geq 0 \quad (\circ)}$$

Problem 11

1. Using the definition of e^{Ft}:

$$e^{Ft} = \sum_{k=0}^{\infty} \frac{t^k}{k!} F^k$$

$$e^{P\tilde{F}P^{-1}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (P\tilde{F}P^{-1})^k$$

Note that

$$(P\tilde{F}P^{-1})^0 = I$$

$$(P\tilde{F}P^{-1})^1 = P\tilde{F}P^{-1}$$

$$(P\tilde{F}P^{-1})^2 = P\tilde{F}\underbrace{P^{-1}P}_{=I}\tilde{F}P^{-1} = P\tilde{F}^2P^{-1}$$

$$(P\tilde{F}P^{-1})^3 = P\tilde{F}\underbrace{P^{-1}P}_{=I}\tilde{F}\underbrace{P^{-1}P}_{=I}\tilde{F}P = P\tilde{F}^3P^{-1}$$

$$\vdots$$

$$(P\tilde{F}P^{-1})^k = P\tilde{F}^kP^{-1}$$

It follows that

$$e^{P\tilde{F}P^{-1}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} P\tilde{F}^kP^{-1}$$

$$= P \sum_{k=0}^{\infty} \frac{t^k}{k!} \tilde{F}^k P^{-1}$$

$$\boxed{e^{P\tilde{F}P^{-1}t} = P e^{\tilde{F}t} P^{-1} \qquad (\boxtimes)}$$

Problem 11:

1. Rewriting equation (o) using ($\boxtimes$) gives

$$x(t) = e^{P\tilde{F}P^{-1}t}\, x(0)$$
$$= P\, e^{\tilde{F}t}\, P^{-1}\, x(0) \quad t \geq 0 \qquad (\nabla)$$

Evaluate $e^{\tilde{F}t}$ as follows

$$e^{\tilde{F}t} = \mathcal{L}^{-1}\left\{ (sI - \tilde{F})^{-1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{pmatrix} s-\lambda_1 & 0 \\ 0 & s-\lambda_2 \end{pmatrix}^{-1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \begin{pmatrix} \frac{1}{s-\lambda_1} & 0 \\ 0 & \frac{1}{s-\lambda_1} \end{pmatrix} \right\}$$

$$e^{\tilde{F}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} u_0(t) \qquad (⧗)$$

Note that $e^{\tilde{F}0} = I$ as expected.