Page 26

1 (a) $z = \cos\alpha + i\sin\alpha = e^{i\alpha}$; $dz = ie^{i\alpha}\,d\alpha$

$$\int_1^{e^{i\theta}} \frac{1}{z}\,dz = \int_0^{\theta} \frac{1}{e^{i\alpha}} \cdot ie^{i\alpha}\,d\alpha = i[\alpha]_0^{\theta}$$

$$= i(\theta - 0) = \underline{i\theta}.\ \text{A}\underline{\text{ns}}$$

— x —

(2) (b) $f = \phi + i\psi$.

where $\phi = \dfrac{x}{x^2+y^2}$

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{x}{x^2+y^2}\right\} = x\,\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right) + \frac{1}{x^2+y^2}$$

$$= -x \cdot \frac{2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$$

$$= \frac{-2x^2 + x^2 + y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

For analyticity:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$

$$\therefore \frac{\partial\psi}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{2y^2-(x^2+y^2)}{(x^2+y^2)^2} = \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{(x^2+y^2)}$$

$$\therefore \psi = \int \frac{2y^2}{(x^2+y^2)^2}\,dy - \int \frac{dy}{x^2+y^2}$$

$$= \int y \cdot \frac{2y}{(x^2+y^2)^2}\,dy - \int \frac{dy}{(x^2+y^2)}$$

$$= y \cdot \left\{-\frac{1}{(x^2+y^2)}\right\} + \int \frac{dy}{x^2+y^2} - \int \frac{dy}{(x^2+y^2)}$$

$$= -\frac{y}{x^2+y^2}$$

Hence the analytic fn. is

$$f = \phi + i\psi = \frac{x}{x^2+y^2} + i\left(-\frac{y}{(x^2+y^2)}\right)$$

$$= \frac{x - iy}{(x^2+y^2)} \quad \underline{\text{Ans}}$$

— x —

(3) we have seen with application of Stokes theorem that

$$\oint_c f(z)\,dz = 2i \int_R \frac{\partial f}{\partial \bar{z}}\,dx\,dy$$

now if $f = \bar{z}$

$$\frac{1}{2i}\oint_c \bar{z}\,dz = \int_R \frac{\partial}{\partial \bar{z}}(\bar{z})\,dx\,dy$$

$$= \int_R dx\,dy = \text{Area of the region included by } c.$$

Proved

— x —

P. T. O.

This I had submitted with previous HW you had graded and asked me to submit with HW #3

PRABIR DARIPA

① $\dfrac{\partial f}{\partial z} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial z} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial z}$ $\qquad x = \dfrac{z+\bar z}{2}, \; y = \dfrac{z-\bar z}{2i}$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right) - i\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}i\right)\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + i\left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)\right\}.$$

$$= \frac{1}{2}\left\{2\cdot\frac{\partial \phi}{\partial x} + i\cdot 2\frac{\partial \psi}{\partial x}\right\}$$

$$= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$ because f is analytic & hence Cauchy Riemann eqs are satisfied

$$\therefore \int_a^b f'(z)\,dz = \int_a^b \left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right)(dx + i\,dy)$$

$$= \int_{z=a}^{z=b}\left(\frac{\partial \phi}{\partial x}\,dx - \frac{\partial \psi}{\partial x}\,dy\right) + i\int_{z=a}^{b}\left(\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \phi}{\partial x}\,dy\right)$$

$$= \int_{z=a}^{z=b}\left(\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\cdot dy\right) + i\int_{z=a}^{b}\left(\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\cdot dy\right)$$

$$= \int_{z=a}^{z=b} d\phi + i\int_{z=a}^{z=b} d\psi$$

$$= \phi + i\psi\Big|_{z=a}^{z=b} = f(z)\Big|_{z=a}^{b} = \underline{f(b) - f(a)}$$

Proved

—x—

② Since $f(z)$ is analytic in D and D is simply connected, $\int_a^z f(\zeta)\,d\zeta$ is independent of path & hence F depends only on z. We have $F = \phi + i\psi$.

$$\therefore \int_a^z f(\zeta)\,d\zeta = \int_a^z (\phi_1 + i\psi_1)(dx + i\,dy_1) = \int_a^z (\phi_1\,dx - \psi_1\,dy) + i\int_a^z (\psi_1\,dx + \phi_1\,dy)$$