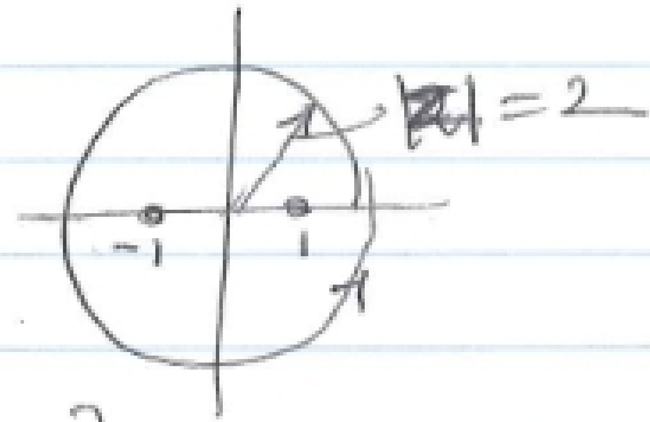
(4)

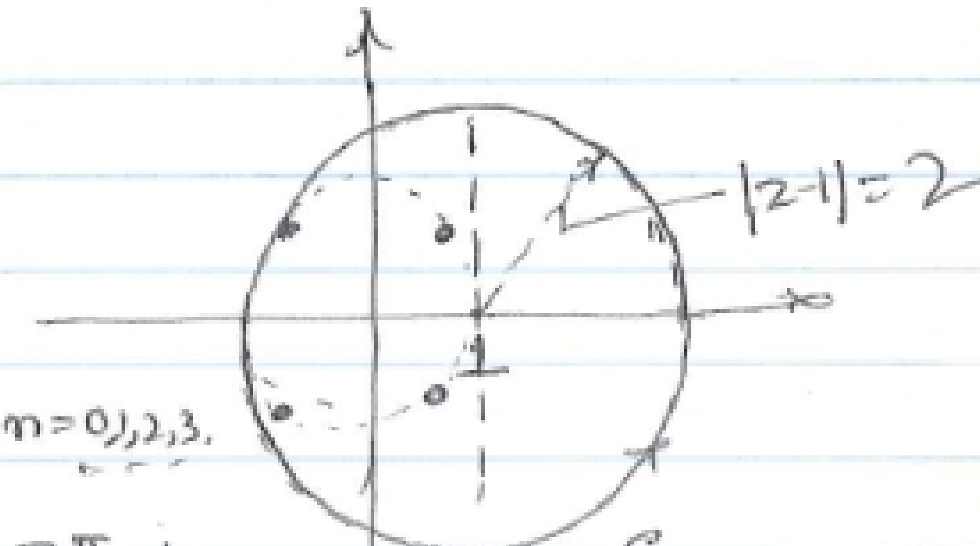
(a) $\oint_{|z|=2} \frac{e^z}{z^2-1}$; poles : $z=\pm 1$



$$= 2\pi i\left\{\text{Res}\left[\frac{e^z}{z^2-1};1\right]+\text{Res}\left[\frac{e^z}{z^2-1};-1\right]\right.$$

$$= 2\pi i\left[\frac{e}{2} - \frac{e^{-1}}{2}\right] = \pi i\left[e - e^{-1}\right] \text{ Ans}$$

(b) $\oint_{|z-1|=2} \frac{dz}{z^4+1}$



$z^4+1 = 0 \Rightarrow z^4 = -1 = e^{(1+2n)\pi i}$, $n=0,1,2,3$.

$z = e^{(1+2n)\pi i/4}$

$= e^{\pi i/4}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi}{4}i}$ - Since all are inside C

$|z-1|=2$; $\oint \frac{dz}{z^4+1} = -\text{Res}\left[f(z); z=\infty\right] = 0$ Ans

$$\oint \frac{dz}{z^4+1} = 2\pi i\left\{\text{Res}\left[\frac{1}{z^4+1}; e^{\pi i/4}\right] + \text{Res}\left[\frac{1}{z^4+1}; e^{\frac{3\pi i}{4}}\right]\right.$$

$$\left. + \text{Res}\left[\frac{1}{1+z^4}; e^{\frac{5\pi i}{4}}\right] + \text{Res}\left[\frac{1}{1+z^4}; e^{\frac{7\pi i}{4}}\right]\right\}$$

$$= 2\pi i\left\{\frac{1}{4e^{3\pi i/4}} + \frac{1}{4e^{9\pi i/4}} + \frac{1}{4e^{15\pi i/4}} + \frac{1}{4e^{21\pi i/4}}\right\}$$

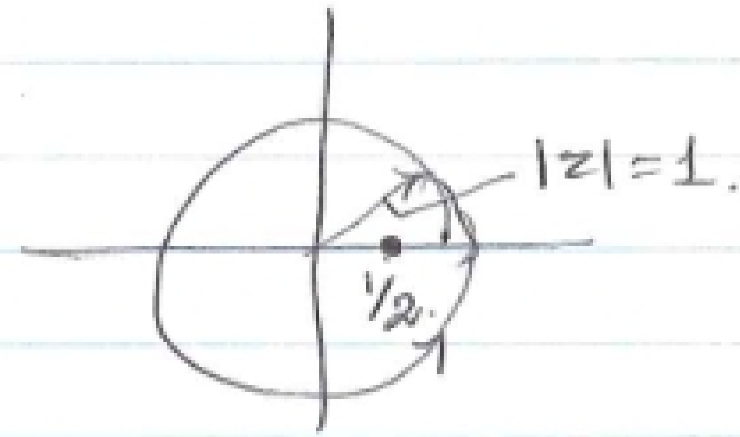
$$= \frac{2\pi i}{4}\left[e^{-3\pi i/4} + e^{-\pi i/4} + e^{-3\pi i/4} + e^{-\pi i/4}\right]$$

$$= \pi i\left[e^{-\pi i/4} + e^{-3\pi i/4}\right] = \pi i\left\{\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}\right\}$$

$= \sqrt{2}\,\pi$ Ans

(c) $I = \oint\limits_{|z|=1} \frac{dz}{2z^2+3z-2} = \oint\limits_{|z|=1} \frac{dz}{(z+2)(2z-1)}$

poles are $z=-2, z=\frac{1}{2}$

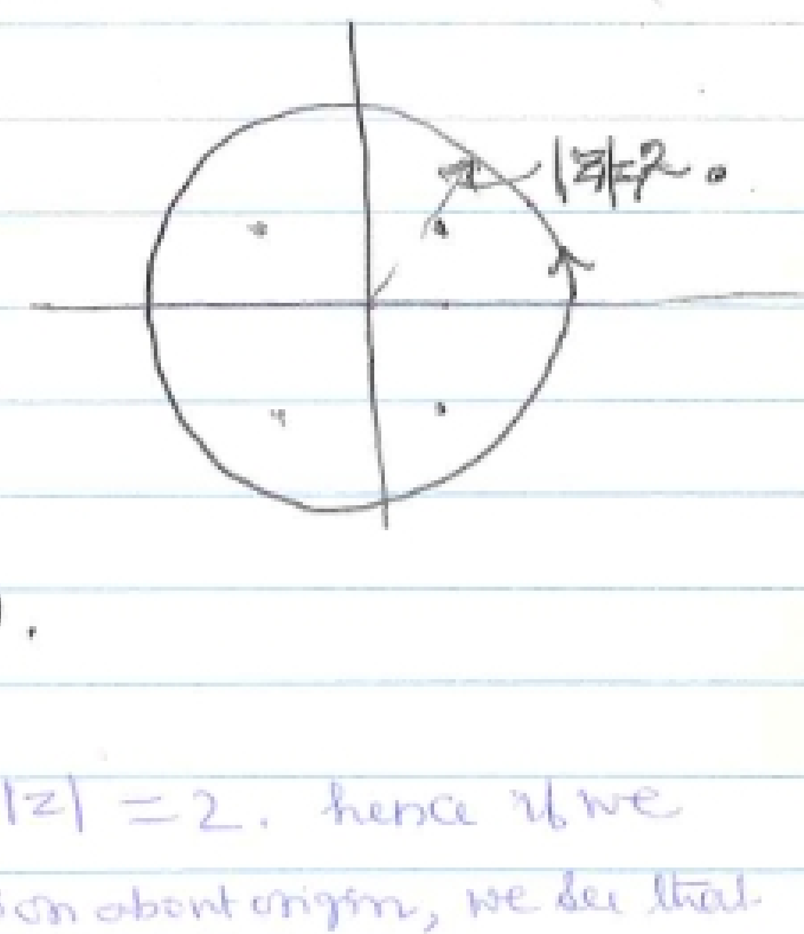


$$I = \oint\limits_{|z|=1} \frac{dz}{(z+2)(2z-1)}$$

$$= \frac{1}{2} \oint\limits_{|z|=1} \frac{dz}{(z+2)(z-1/2)} = \frac{1}{2}\left[\frac{2\pi i}{z+2}\right]_{z=\frac{1}{2}}$$

$$= \frac{1}{2} \times \frac{2\pi i}{\frac{1}{2}+2} = \frac{2\pi i}{5} \text{ Ans.}$$

(d) $\oint\limits_{|z|=2} \frac{z^3+2z}{z^4+z^2+2} dz = \oint\limits_{|z|=2} \frac{z(z^2+2)}{z^4+z^2+2} = I \text{ (say)}.$



$z^4+z^2+2=0$

$\Rightarrow z^2 = \frac{-1 \pm \sqrt{1-4.1.2}}{2.1.}$

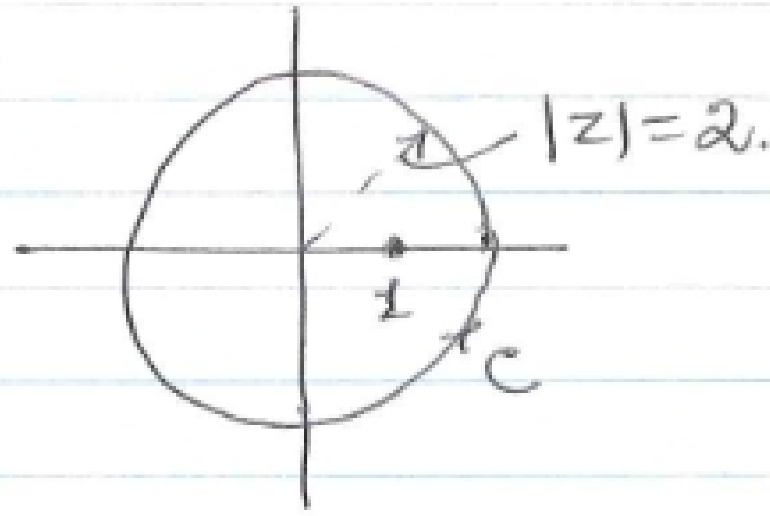
$= \frac{-1 \pm \sqrt{-7}}{2.1} = \frac{-1 \pm \sqrt{7}i}{2.}$

$z^2 = \left(-\frac{1}{2} + \frac{\sqrt{7}}{2}i\right) \& \left(-\frac{1}{2} - \frac{\sqrt{7}}{2}i\right).$

So $I = \oint\limits_{|z|=2} \frac{(z^3+2z)dz}{(z^4+z^2+2)}$, $f(z)$ has all pole inside $|z|=2$. hence if we do laurent Expansion about origin, we see that the first term will be $\frac{1}{z}$ & then power of z will increase in denominator.

$$= \oint\limits_{2<|z|} \frac{dz}{z} + \oint\limits_{2<|z|} \frac{()}{z^2} + \cdots = \oint\limits_{2<|z|} \frac{dz}{z} + 0 \cdots = 2\pi i \text{ Ans}$$

4 (e) $\displaystyle\oint_{|z|=2} \frac{dz}{(z-1)^4(z-4)}$



$$= \oint \frac{\left(\frac{1}{z-4}\right)}{(z-1)^4}\cdot dz$$

$$= \frac{2\pi i}{3!} \times \left\{ \frac{3!}{2\pi i} \oint \frac{\left(\frac{1}{z-4}\right)}{(z-1)^{3+1}} dz \right\}$$

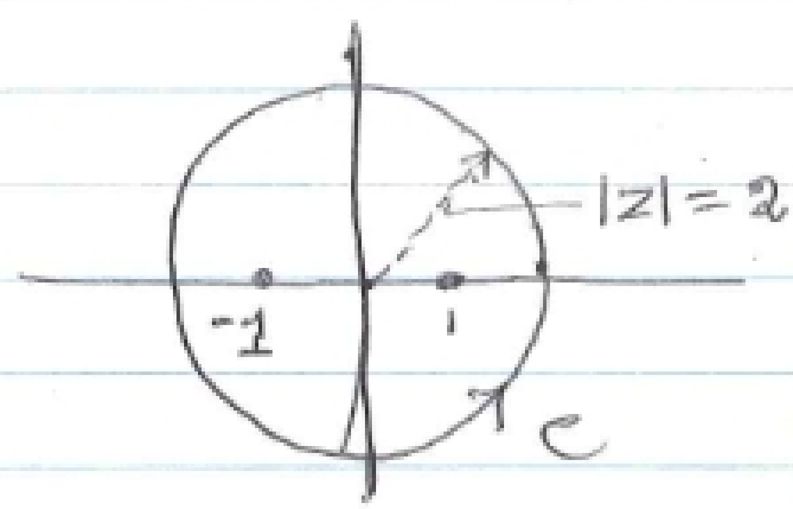
$$= \frac{2\pi i}{3!} \frac{d^3}{dz^3}\left(\frac{1}{z-4}\right)\Big|_{z=1} = \frac{2\pi i}{3!} \times (-1\cdot -2 \cdot -3) \frac{1}{(z-4)^4}\Big|_{z=1}$$

$$= -2\pi i \times \frac{1}{(-3)^4} = -\frac{2\pi i}{81} \quad \underline{\underline{\text{Ans}}}.$$

— x —

4 (f)

$$\oint_{|z|=2} \frac{\sin z}{(z^2-1)(z^2-9)} dz :$$



$$= 2\pi i \left\{ \text{Res}\left[\frac{\sin z}{(z^2-1)(z^2-9)} ; z=1\right] + \text{Res}\left[\frac{\sin z}{(z^2-1)(z^2-9)} ; z=-1\right] \right\}$$

$$= 2\pi i \left\{ \left[\frac{\sin z}{(z^2-9)(z+1)}\right]_{z=1} + \left[\frac{\sin z}{(z^2-9)(z-1)}\right]_{z=-1} \right\}.$$

using the theorem in Prob 2(a)

$$= 2\pi i \left[\frac{\sin 1}{-8\times 2} + \frac{\sin(-1)}{-8\times(-2)} \right]$$

$$= -\frac{2\pi i}{16}\left[\sin 1 + \sin 1\right] = \left(-\frac{\pi i}{\cancel{8}\,4} \sin 1\right) \quad \underline{\underline{\text{Ans}}}$$