5-3 Inversion of Rational Functions

All the Laplace Transform you will encounter has the following form:

$$\frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots a_{n-1}s + a_n} e^{-Ts}$$

Rational function X(s) Delay

Why? Rational functions come out naturally from the Laplace transform of ordinary differential equations, exponential, cosine and sine functions.

Our strategy: *breakdown a general rational function into simpler fractions and polynomials whose inverse transforms have already been computed in Table 5-3.*

How?

I. Convert non-proper rational function into proper rational function
Non-proper: degree of numerator ≥ degree of denominator

Approach: Long Division
Example:

$$X(s) = \frac{s^5 + 5s^4 + 9s^3 + 9s^2 + 12s + 4}{s^4 + 4s^3 + 5s^2 + 4s + 4}$$

$$\begin{array}{r}
s+1 \\
s^4+4s^3+5s^2+4s+4 \overline{\big) s^5+5s^4+9s^3+9s^2+12s+4} \\
s^5+4s^4+5s^3+4s^2+4s \\
\hline
s^4+4s^3+5s^2+8s+4 \\
s^4+4s^3+5s^2+4s+4 \\
\hline
4s
\end{array}$$

$$X(s) = s + 1 + \frac{4s}{s^4 + 4s^3 + 5s^2 + 4s + 4}$$

Polynomial Proper fraction

It is easy to find the Laplace transform of polynomials:

$$s^n \leftrightarrow \delta^{(n)}(t)$$

$$L^{-1}(s+1) = L^{-1}(s) + L^{-1}(1) = \delta^{(1)}(t) + \delta(t)$$

II. Factorize the denominator polynomial:

Example:

$$s^4 + 4s^3 + 5s^2 + 4s + 4 = (s+2)^2(s^2+1) = (s+2)^2(s+j)(s-j)$$

- Factorize into REAL linear and irreducible quadratic factors. Further break down REAL irreducible quadratic factors into conjugate roots.
- Notice some factors may repeat.
- No analytical formula for polynomial with degree 5 or other. Need to rely on numerical methods.

Four simple rules to write down unknowns:

1. $\frac{\cdots}{(s-a)\ldots} \to \frac{k}{s-a}+\ldots$ for real root a
2. $\frac{\cdots}{(s-a)(s-\bar{a})\ldots} \to \frac{k}{s-a}+\frac{\bar{k}}{s-\bar{a}}+\ldots$ for complex conjugate roots a and $\bar{a}$
3. $\frac{\cdots}{(s-a)^n\ldots} \to \frac{k_1}{s-a}+\frac{k_2}{(s-a)^2}+\ldots+\frac{k_n}{(s-a)^n}+\ldots$
4. $\frac{\cdots}{(s-a)^n(s-\bar{a})^n\ldots} \to \frac{k_1}{s-a}+\frac{\bar{k_1}}{s-\bar{a}}+\ldots+\frac{k_n}{(s-a)^n}+.\frac{\bar{k_n}}{(s-a)^n}+\ldots$

IV. Solve for the unknowns.

Many methods exist. I will talk about two:

1. Compare coefficients (The Dumb Way)

Example: $X(s)=\frac{4s}{(s+2)^2(s^2+1)}$

$$
\begin{aligned}
X(s) &= \frac{a}{s+2}+\frac{b}{(s+2)^2}+\frac{d}{s+j}+\frac{\bar{d}}{s-j} \\
&= \frac{(a+2\,\mathrm{Re}(d))s^3+(2a+b+8\,\mathrm{Re}(d)+2\,\mathrm{Im}(d))s^2+(a+8\,\mathrm{Re}(d)+8\,\mathrm{Im}(d))s+(2a+b+8\,\mathrm{Im}(d))}{(s+2)^2(s^2+1)}
\end{aligned}
$$

Notice that when you expand it out, no complex coefficients remain and you can ALWAYS do that if you follow the four simple rules above.

You can ask Matlab to do this too: