

5-3 Inversion of Rational Functions



All the Laplace Transform you will encounter has the following form:

$$\frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n} e^{-Ts}$$

$7 \frac{dy}{dt^2} + 3 \frac{dy}{dt} + 4y(t) = 6 \frac{dx}{dt} + x(t)$
 $\downarrow \mathcal{L}$ (Initial condition zero)
 $s^2 Y(s) + 3s Y(s) + 4Y(s) = 6s X(s) + X(s)$

$H(s) = \frac{Y(s)}{X(s)} = \frac{6s+1}{7s^2+3s+4}$

Rational function $X(s)$ $\xrightarrow{\mathcal{L}^{-1}}$ $x(t)$ $\xrightarrow{\text{Delay}}$ $x(t-T)$

Focus on the "how" \rightarrow $x(t)$

Why? Rational functions come out naturally from the Laplace transform of ordinary differential equations, exponential, cosine and sine functions.

$X(s) = 1/s$
 $H(s)X(s) = Y(s)$

Our strategy: *breakdown a general rational function into simpler fractions and polynomials whose inverse transforms have already been computed in Table 5-3.*

Use transfer function to compute the output given a specific input.

How?

$$Y(s) = \frac{6s+1}{7s^2+3s+4} \cdot \frac{1}{s} = \frac{6s+1}{7s^3+3s^2+4s}$$

- I. Convert non-proper rational function into proper rational function
- Non-proper: degree of numerator \geq degree of denominator

Approach: Long Division

Example:

$$X(s) = \frac{s^5 + 5s^4 + 9s^3 + 9s^2 + 12s + 4}{s^4 + 4s^3 + 5s^2 + 4s + 4}$$

$\alpha = \frac{10}{3}$ non-proper
 $3 \frac{1}{3}$ proper
 $3 \overline{)10}$
 $\quad 9$
 $\quad \underline{1}$

$X(s) = s + 1 + \frac{4s}{s^4 + 4s^3 + 5s^2 + 4s + 4}$

Polynomial: $s + 1$
 Proper fraction: $\frac{4s}{s^4 + 4s^3 + 5s^2 + 4s + 4}$

Reason: partial fraction for proper fraction only
 Inverse Laplace transfer is easy

Long division steps:
 $s^4 + 4s^3 + 5s^2 + 4s + 4 \overline{) s^5 + 5s^4 + 9s^3 + 9s^2 + 12s + 4}$
 $\underline{s^4 + 4s^3 + 5s^2 + 4s}$
 $s^4 + 4s^3 + 5s^2 + 8s + 4$
 $\underline{s^4 + 4s^3 + 5s^2 + 4s}$
 $4s$

Annotations:
 - $s+1$ is circled in red.
 - "cancel out the leading power" points to the subtraction step.
 - "non-proper" and "proper" labels are in red.
 - "lead quotient" points to the $3 \frac{1}{3}$ calculation.

Rarely occur in real analysis $\Rightarrow \mathcal{L}^{-1}[1] = \delta(t)$ $\mathcal{L}^{-1}[s] = \frac{d\delta}{dt}$ $\mathcal{L}^{-1}[s^2] = \frac{d^2\delta}{dt^2}$

It is easy to find the Laplace transform of polynomials:

$$s^n \leftrightarrow \delta^{(n)}(t)$$

$$L^{-1}(s+1) = L^{-1}(s) + L^{-1}(1) = \delta^{(1)}(t) + \delta(t)$$

II. Factorize the **denominator** polynomial:

Example:

$$s^4 + 4s^3 + 5s^2 + 4s + 4 = (s+2)^2(s^2+1) = (s+2)^2(s+j)(s-j)$$

- Factorize into REAL linear and irreducible quadratic factors. Further break down REAL irreducible quadratic factors into conjugate roots.
- Notice some factors may repeat.
- No analytical formula for polynomial with degree 5 or other. Need to rely on numerical methods.

Four simple rules to write down unknowns:

1. $\frac{\dots}{(s-a)\dots} \rightarrow \frac{k}{s-a} + \dots$ for real root a
2. $\frac{\dots}{(s-a)(s-\bar{a})\dots} \rightarrow \frac{k}{s-a} + \frac{\bar{k}}{s-\bar{a}} + \dots$ for complex conjugate roots a and \bar{a}
3. $\frac{\dots}{(s-a)^n \dots} \rightarrow \frac{k_1}{s-a} + \frac{k_2}{(s-a)^2} + \dots + \frac{k_n}{(s-a)^n} + \dots$
4. $\frac{\dots}{(s-a)^n (s-\bar{a})^n \dots} \rightarrow \frac{k_1}{s-a} + \frac{\bar{k}_1}{s-\bar{a}} + \dots + \frac{k_n}{(s-a)^n} + \frac{\bar{k}_n}{(s-\bar{a})^n} + \dots$

IV. Solve for the unknowns.

Many methods exist. I will talk about two:

1. Compare coefficients (The Dumb Way)

Example: $X(s) = \frac{4s}{(s+2)^2(s^2+1)}$

$$\begin{aligned} X(s) &= \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{d}{s+j} + \frac{\bar{d}}{s-j} \\ &= \frac{(a+2\operatorname{Re}(d))s^3 + (2a+b+8\operatorname{Re}(d)+2\operatorname{Im}(d'))s^2 + (a+8\operatorname{Re}(d)+8\operatorname{Im}(d))s + (2a+b+8\operatorname{Im}(d))}{(s+2)^2(s^2+1)} \end{aligned}$$

Notice that when you expand it out, no complex coefficients remain and you can ALWAYS do that if you follow the four simple rules above.

You can ask Matlab to do this too: