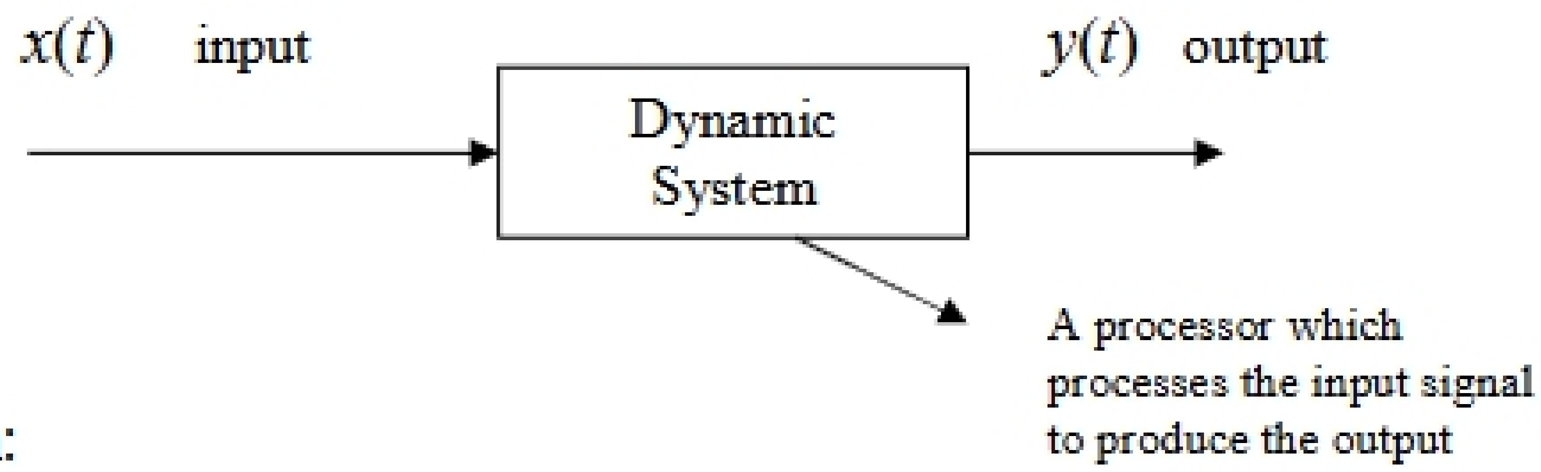
Chapter 5 The Laplace Transform

5-1 Introduction

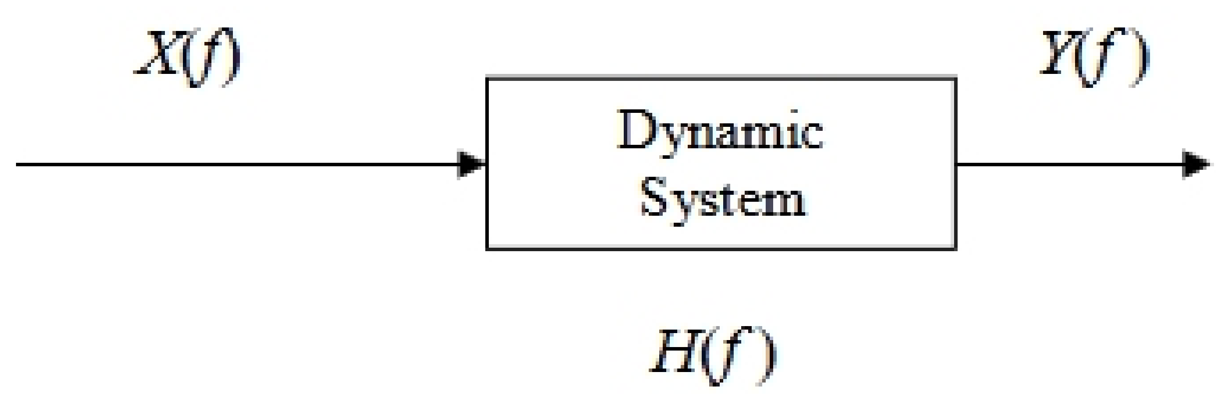
(1) System analysis



Linear Dynamic system:

$$\frac{dy^{(n)}(t)}{dt^n} + a_1 \frac{dy^{(n-1)}(t)}{dt^{n-1}} + \ldots + a_n y(t) = b_0 \frac{dx^{(m)}(t)}{dt^m} + \ldots + b_m x(t)$$

Question: Can we determine *y(t)* for a given *x(t)*?
Answer: Use Fourier transform to convert the ODE into algebraic equation!



$$Y(f) = H(f)X(f)$$
$$y(t) = F^{-1}(Y(f))$$

(2) Two Problems

1. Some common signals do not have a Fourier Transform !

   Example: What is the Fourier transform of $x(t) = e^t u(t)$?

   $X(f) = \int_{-\infty}^{\infty} e^t u(t) e^{-j2\pi ft} dt = \int_{0}^{\infty} e^t e^{-j2\pi ft} dt$ does not exist for any f as e^t blows up when $t \to \infty$.

   Even though $x(t)$ grows unbounded as $t \to \infty$, it may still exist as an intermediate step in a larger system. Consider the output when $x(t)$ is fed into a system with impulse response $h(t) = \left(3e^{-2t} - 2e^{-t}\right)u(t)$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{t-\tau}u(t-\tau)\left(3e^{-2\tau}-2e^{-\tau}\right)u(\tau)d\tau \\ &= \int_{0}^{t} e^{t-\tau}\left(3e^{-2\tau}-2e^{-\tau}\right)d\tau \\ &= \left(e^{-t}-e^{-2t}\right)u(t) \end{aligned}$$

which certainly decays to 0 as $t \to \infty$.

2. Initial Condition Problem
   Say we know the output y(t) of the above dynamic system is 5 at t=0. Nowhere in the Fourier system equations below we could insert this information:

$$Y(f) = H(f)X(f)$$
$$y(t) = F^{-1}(Y(f))$$

(3) Solution: Laplace Transform

Even though $x(t)$ does not go to zero (when $t \to \infty$), but $x(t)e^{-\sigma t}$ may for large enough α.

- ⇨ We will assume all signals are "causal" : $x(t) = 0$ for $t < 0$
- ⇨ Fourier transform of $x(t)e^{-\sigma t}$:

$$\begin{aligned} \int_0^{\infty} (x(t)e^{-\sigma t})e^{-j\omega t}dt &= \int_0^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\ &= \int_0^{\infty} x(t)e^{-st}dt \quad \text{where } s = \sigma + j\omega \\ &= \text{Laplace Transform of } x(t) \end{aligned}$$

- We will see how Laplace transform takes care of initial conditions later.
- Even though inverse Laplace Transform exists, it involves more sophisticated concepts from complex number theory. Just like Inverse Fourier Transform, we will just use table (and Matlab).
- Laplace transform is just as nice as Fourier Transform:

$$Y(s) = H(s)X(s) + \textit{initial conditions}$$
$$y(t) = L^{-1}(Y(s))$$

- Fourier Transform of $x(t)u(t)$ can obtained by substituting $s = j\omega$ (i.e. setting $\sigma = 0$) in $X(s)$.[1].

[1] If the Region of Converge of X(s) does not include the imaginary axis ($s = j\omega$), then its Fourier Transform does not exist. (More later)

5-2 Examples of Evaluating Laplace Transforms using the definition

(1) Step function $x(t)=u(t)$

$$
\begin{aligned}
L[u(t)] &= \int_0^\infty e^{-st}dt \\
&= -\frac{1}{s}\int_0^\infty e^{-st}d(-st) \\
&= \left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=\infty} \\
&= -\frac{1}{s}\lim_{t\to\infty}\left(e^{-st}\right)+\frac{1}{s} \\
&= -\frac{1}{s}\lim_{t\to\infty}\left(e^{-\mathrm{Re}(s)t}e^{-j\,\mathrm{Im}(s)t}\right)+\frac{1}{s} \\
&= \begin{cases} 1/s & \text{if } \mathrm{Re}(s)>0 \text{ as } e^{-\mathrm{Re}(s)t} \ \square^{t}\square \to 0 \\ \infty & \text{if } \mathrm{Re}(s)<0 \text{ as } e^{-\mathrm{Re}(s)t} \ \square^{t}\square \to \infty \\ \text{not sure} & \text{if } \mathrm{Re}(s)=0 \text{ as } \lim_{t\to\infty} e^{-j\,\mathrm{Im}(s)t} = ? \end{cases}
\end{aligned}
$$

When Re(s) = 0, define ω=Im(s). The integral becomes $\int_0^\infty e^{-j\omega t}dt = \int_{-\infty}^\infty u(t)e^{-j\omega t}dt$ which is the Fourier transform of u(t). From chapter 4, we know that

$$F[u(t)] = \frac{1}{j\omega} + 2\pi\delta(\omega)$$

Note that for ω≠0, the Fourier Transform can be evaluated by substituting s=jω in the expression 1/s.

(2) Exponential $x(t) = e^{-at}u(t)$

$$
\begin{aligned}
L[e^{-at}u(t)] &= \int_0^\infty e^{-at}e^{-st}dt \\
&= \int_0^\infty e^{-(s+a)t}dt \\
&= \frac{-1}{s+a}\int_0^\infty e^{-(s+a)t}d-(s+a)t \\
&= \frac{-1}{s+a}\lim_{t\to\infty} e^{-(s+a)t} + \frac{1}{s+a} \\
&= -\frac{1}{s+a}\lim_{t\to\infty}\left(e^{-\mathrm{Re}(s+a)t}e^{-j\,\mathrm{Im}(s+a)t}\right)+\frac{1}{s+a} \\
&= \begin{cases} 1/(s+a) & \text{if } \mathrm{Re}(s)>-\mathrm{Re}(a) \text{ as } e^{-\mathrm{Re}(s+a)t} \ \square^{t}\square \to 0 \\ \infty & \text{if } \mathrm{Re}(s)<-\mathrm{Re}(a) \text{ as } e^{-\mathrm{Re}(s+a)t} \ \square^{t}\square \to \infty \\ 1/(s+a)+2\pi\delta(\mathrm{Im}(s)) & \text{if } \mathrm{Re}(s)=-\mathrm{Re}(a) \text{ as } \lim_{t\to\infty} e^{-j\,\mathrm{Im}(s+a)t} = ? \end{cases}
\end{aligned}
$$