ME451: Control Systems

Lecture 2
Laplace transform

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Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis

- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - (Nyquist)

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations &) laboratories

Laplace transform

- One of most important math tools in the course!
- Definition: For a function f(t) (f(t)=0 for t<0),

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt$$

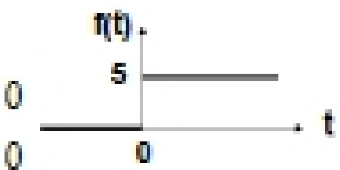
(s: complex variable)

- We denote Laplace transform of f(t) by F(s).

Example of Laplace transform

- Step function

$$f(t) = 5u(t) = \begin{cases} 5 & t \ge 0 \\ 0 & t < 0 \end{cases}$$



$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 5e^{-st}dt = 5\int_0^\infty e^{-st}dt$$

$$= 5\left[-\frac{1}{s}\left[e^{-st}\right]_0^\infty\right] = 5\left[\frac{1}{s}\right] = \frac{5}{s}$$

Remember $\mathcal{L}(u(t)) = 1/s$

Integration is Hard

Tables are Easier

Laplace transform table

(Table B.1 in Appendix B of the textbook)

	Time Function $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$, $s > 0$
2	t (unit-ramp function)	$\frac{1}{s^2}$, $s > 0$
3	t^n (n, a positive integer)	$\frac{n!}{s^{n+1}}$, $s > 0$
4	e^{at}	$\frac{1}{s-a}$, $s > a$
5	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$, $s > 0$
6	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$, $s > 0$
7	$t^n g(t)$, for $n = 1, 2, \ldots$	$(-1)^n \frac{d^n G(s)}{ds^n}$
8	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$, $s > \lvert\omega\rvert$
9	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$, $s > \lvert\omega\rvert$
10	$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$ Scale property
11	$e^{at}g(t)$	$G(s-a)$ Shift property
12	$e^{at}t^n$, for $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$

$\mathcal{L}$ $\quad$ $\mathcal{L}^{-1}$

Inverse Laplace Transform

Properties of Laplace transform

Linearity

$$\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

Proof.

$$\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \int_0^\infty (\alpha_1 f_1(t) + \alpha_2 f_2(t))e^{-st}dt = \alpha_1 \underbrace{\int_0^\infty f_1(t)e^{-st}dt}_{F_1(s)} + \alpha_2 \underbrace{\int_0^\infty f_2(t)e^{-st}dt}_{F_2(s)}$$

Ex.

$$\mathcal{L}\left\{5u_s(t) + 3e^{-2t}\right\} = 5\mathcal{L}\{u_s(t)\} + 3\mathcal{L}\left\{e^{-2t}\right\} = \frac{5}{s} + \frac{3}{s+2}$$

Properties of Laplace transform

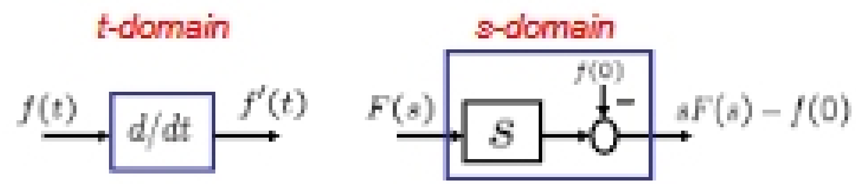
Differentiation

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

Proof.

$$\mathcal{L}\left\{f'(t)\right\} = \int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

Ex. $\mathcal{L}\left\{(\cos 2t)'\right\} = s\mathcal{L}\{\cos 2t\} - 1 = \frac{s^2}{s^2+4} - 1 = \frac{-4}{s^2+4} (= \mathcal{L}\{-2\sin 2t\})$



Properties of Laplace transform

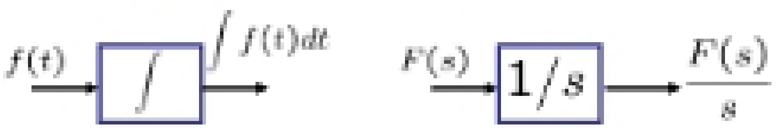
Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof. $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \left(\int_0^t f(\tau)d\tau\right)e^{-st}dt$

$= -\frac{1}{s}\left[\left(\int_0^t f(\tau)d\tau\right)e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt = \frac{F(s)}{s}$

t-domain *s-domain*



Properties of Laplace transform

Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$ if all the poles of sF(s) are in the left half plane (LHP)

Ex. $F(s) = \frac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0}\frac{5}{s^2+s+2} = \frac{5}{2}$

Poles of sF(s) are in LHP, so final value thm applies.

Ex. $F(s) = \frac{4}{s^2+4} \Rightarrow \lim_{t\to\infty} f(t) \neq \lim_{s\to 0}\frac{4s}{s^2+4} = 0$

Some poles of sF(s) are not in LHP, so final value thm does NOT apply.

Properties of Laplace transform

Initial value theorem

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$ if the limits exist.

Remark: In this theorem, it does not matter if pole location is in LHP or not.

Ex. $F(s) = \frac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Ex. $F(s) = \frac{4}{s^2+4} \Rightarrow \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Properties of Laplace transform

Convolution

$$\left.\begin{array}{l} F_1(s) = \mathcal{L}\{f_1(t)\} \\ F_2(s) = \mathcal{L}\{f_2(t)\} \end{array}\right\}$$

$$\Rightarrow \quad F_1(s)F_2(s) = \mathcal{L}\Big\{\overbrace{\int_0^t f_1(\tau)f_2(t-\tau)d\tau}^{\text{Convolution}}\Big\} = \mathcal{L}\left\{\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right\}$$

IMPORTANT REMARK

$$F_1(s)F_2(s) \neq \mathcal{L}\{f_1(t)f_2(t)\}$$

$$\mathcal{L}^{-1}(F_1(s)F_2(s)) \neq f_1(t)f_2(t)$$