Gauss' Law

Disclaimer: These lecture notes are not meant to replace the course textbook. The content may be incomplete. Some topics may be unclear. These notes are only meant to be a study aid and a supplement to your own notes. Please report any inaccuracies to the professor.

Preamble

In this lecture we learn a simple and powerful technique for calculating the electric field for situations involving a high degree of symmetry.

Flux

Flux, Φ, is a measure of the amount of a vector field, $\mathbf{F}$, passing through a surface with area $\mathbf{A}$ (direction defined perpendicular to the plane of the surface):

$$\Phi = \mathbf{F}\cdot\mathbf{A} = FA\cos\theta$$

where θ is the angle between the field direction and the normal to the area surface.

As one example, consider the flow of air or water through a surface. The voulume of ar passing through the surface per unit time is given by:

$$\Phi = \mathbf{v}\cdot\mathbf{A} = vA\cos\theta$$

Obviously this flow rate depends on the orientation of the surface.

As another example, consider the light from the Sun impinging onto the surface of the Earth. Obviously land near the equator receives a large light flux than that at the poles.

In this course, we will consider the flux of electric and magnetic fields (essentially light!) passing through a surface.

Flux Through a Closed Surface

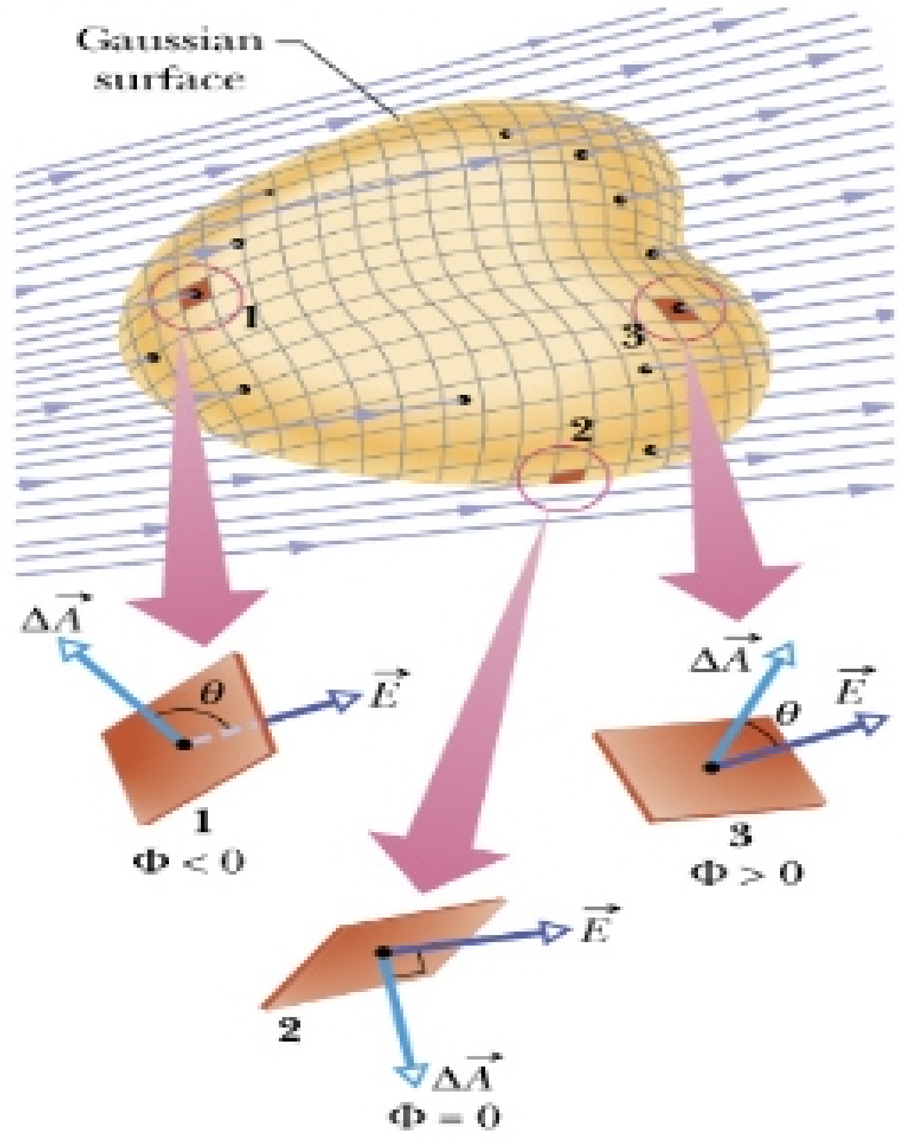


Fig. from HRW 7/e

Consider the flux of electric field through a closed surface. Let's break the surface into a large number (N) of small elements of area $\Delta\mathbf{A}$ that are nearly flat. By convention, let's take the direction of the area vector to be perpendicular to the surface and pointing out.

The flux of electric field passing through this closed surface is then:

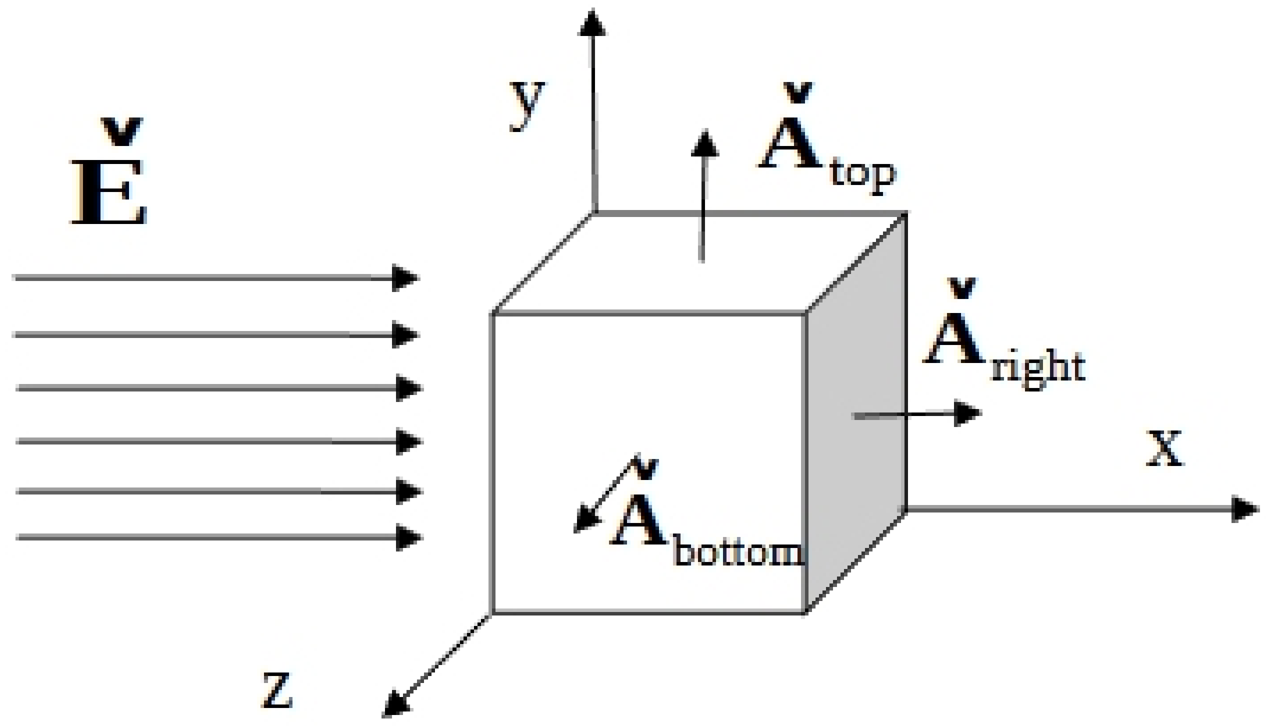
$$\Phi = \sum_{i=1}^{N} \mathbf{E} \cdot \Delta\mathbf{A}$$

If we let $N \to \infty$ and $\Delta\mathbf{A} \to 0$, then the small elements become infinitesimal and we can write:

$$\Phi = \oint \mathbf{E} \cdot d\mathbf{A}$$

This is a closed surface integral. It is a 2-dimensional integral over a 3-dimensional surface. The "circle" in the integral sign denotes an integral over an entire closed surface (versus just a single face).

Example



Consider a square cube immersed in a constant electric field: $\mathbf{E}=E_0\hat{\mathbf{x}}$. Calculate the flux through each face.

$\vec{\mathbf{A}}_i$ is the area vector pointing out from cube face *i*, which has magnitude $|\vec{\mathbf{A}}_i|=s^2$, where *s* is the side length of the cube.

Note that $\vec{\mathbf{E}}\cdot\vec{\mathbf{A}}_{top}=0$ since $\vec{\mathbf{E}}\perp\vec{\mathbf{A}}_{top}$

$\vec{\mathbf{E}}\cdot\vec{\mathbf{A}}_{bottom}=0$

Also: $\vec{\mathbf{E}}\cdot\vec{\mathbf{A}}_{front}=0$

$\vec{\mathbf{E}}\cdot\vec{\mathbf{A}}_{back}=0$

What is left is:

$\vec{\mathbf{E}}\cdot\vec{\mathbf{A}}_{right}=|\vec{\mathbf{E}}||\vec{\mathbf{A}}_{right}|=E_0s^2$ since $\vec{\mathbf{E}}\parallel\vec{\mathbf{A}}_{right}$

$\vec{\mathbf{E}}\cdot\vec{\mathbf{A}}_{left}=-|\vec{\mathbf{E}}||\vec{\mathbf{A}}_{left}|=-E_0s^2$ since $\vec{\mathbf{E}}$ opposite $\vec{\mathbf{A}}_{left}$

$\Phi=\sum_{i=1}^{6}\mathbf{E}\cdot\mathbf{A}_i=E_0s^2-E_0s^2=0$ sum over all 6 faces

Thus, the total flux into the cube is balanced by the flux going out. This will hold for any arbitrary shape of the surface, provided the electric field is constant