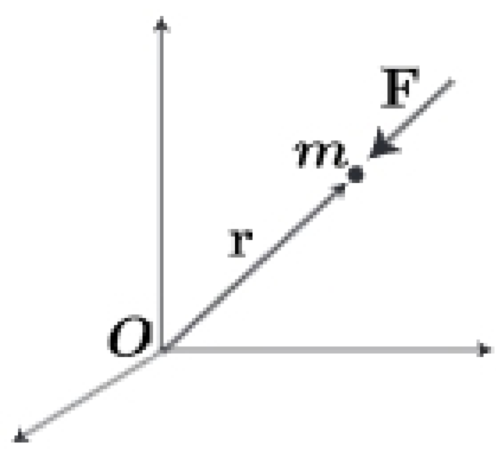
J. Peraire, S. Widnall
16.07 Dynamics
Fall 2008
Version 1.2

Lecture L15 - Central Force Motion: Kepler's Laws

When the only force acting on a particle is always directed towards a fixed point, the motion is called *central force motion*. This type of motion is particularly relevant when studying the orbital movement of planets and satellites. The laws which govern this motion were first postulated by Kepler and deduced from observation. In this lecture, we will see that these laws are a consequence of Newton's second law. An understanding of central force motion is necessary for the design of satellites and space vehicles.

Kepler's Problem

We consider the motion of a particle of mass m, in an inertial reference frame, under the influence of a force, $\boldsymbol{F}$, directed towards the origin.



We will be particularly interested in the case when the force is inversely proportional to the square of the distance between the particle and the origin, such as the gravitational force. In this case,

$$\boldsymbol{F} = -\frac{\mu}{r^2} m \boldsymbol{e}_r ,$$

where μ is the gravitational parameter, r is the modulus of the position vector, $\boldsymbol{r}$, and $\boldsymbol{e}_r = \boldsymbol{r}/r$.

It can be shown that, in general, Kepler's problem is equivalent to the two-body problem, in which two masses, M and m, move solely due to the influence of their mutual gravitational attraction. This equivalence is obvious when $M \gg m$, since, in this case, the center of mass of the system can be taken to be at M.

However, even in the more general case when the two masses are of similar size, we shall show that the problem can be reduced to a "Kepler" problem.
Although most problems in celestial mechanics involve more than two bodies, many problems of practical interest can be accurately solved by just looking at two bodies at a time. When more than two bodies are involved, the problem is considerably more complicated, and, in this case, no general solutions are known.
The two body problem was studied by Kepler (1571-1630) who lived before Newton was born. His interest was in describing the motion of planets around the sun. He postulated the following laws:

1.- The orbits of the planets are ellipses with the Sun at one focus

2.- The line joining a planet to the Sun sweeps out equal areas in equal intervals of time

3.- The square of the period of a planet is proportional to the cube of the major axis of its elliptical orbit

In this lecture, we will start from Newton's laws and verify that the above three laws can indeed be derived from Newtonian mechanics.

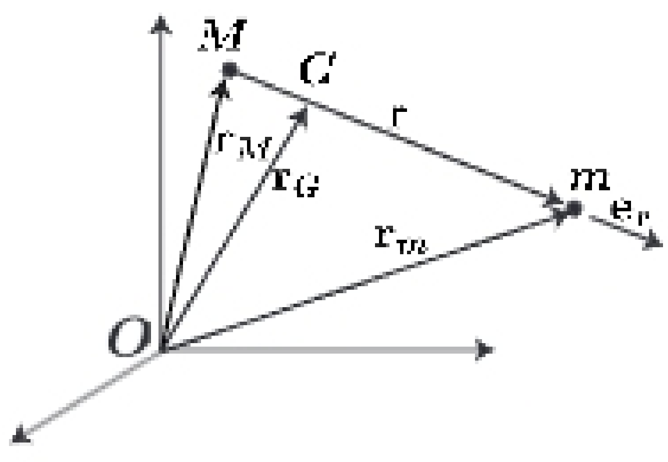
Equivalence between the two-body problem and Kepler's problem

Here we consider the problem of two isolated bodies of masses M and m which interact though gravitational attraction. Let $\boldsymbol{r}_M$ and $\boldsymbol{r}_m$ denote the position vectors of the two bodies relative to a fixed origin O. Since the only force acting on the bodies is the force of mutual gravitational attraction, the motion is governed by Newton's law with an equal and opposite force acting on each body.

$$M\ddot{\boldsymbol{r}}_M = G\frac{Mm}{r^2}\boldsymbol{e}_r, \tag{1}$$

$$m\ddot{\boldsymbol{r}}_m = -G\frac{Mm}{r^2}\boldsymbol{e}_r\ , \tag{2}$$

where $r = |\boldsymbol{r}|$, $\boldsymbol{e}_r = \boldsymbol{r}/r$, and G is the gravitational constant.



The position of the center of gravity, G, of the two bodies will be

$$\boldsymbol{r}_G = \frac{M\boldsymbol{r}_M + m\boldsymbol{r}_m}{M+m}\ . \tag{3}$$

Since the two bodies are isolated, we will have, from momentum conservation, that $\dot{\boldsymbol{r}}_G$ =*constant*, and $\ddot{\boldsymbol{r}}_G = \boldsymbol{0}$. Therefore, the position of the center of gravity, at all times, can be found trivially from the initial conditions.

If the position vector of m as observed by M, $\boldsymbol{r} = \boldsymbol{r}_m - \boldsymbol{r}_M$, is known, then the position vectors of M and m could be computed as

$$\boldsymbol{r}_M = \boldsymbol{r}_G - \frac{m}{M+m}\boldsymbol{r}, \qquad \boldsymbol{r}_m = \boldsymbol{r}_G + \frac{M}{M+m}\boldsymbol{r} \ . \tag{4}$$

Therefore, since we know the position of the center of mass r_G for all time, we shall show that the problem of determining $\boldsymbol{r}_M$ and $\boldsymbol{r}_m$ is equivalent to that of determining $\boldsymbol{r}$, the vector distance between them.

The governing equations for r_m and r_M are given in equation (1) and (2). Subtracting these two expressions, we obtain,

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_m - \ddot{\boldsymbol{r}}_M = -G\frac{M+m}{r^2}\,\boldsymbol{e}_r \ , \tag{5}$$

or,

$$\frac{Mm}{M+m}\ddot{\boldsymbol{r}} = -G\frac{Mm}{r^2}\,\boldsymbol{e}_r \ . \tag{6}$$

The above expression shows that the motion of m relative to M is in fact a Kepler problem in which the force is given by $-GMm\boldsymbol{e}_r/r^2$ (this is indeed the real force), but the mass of the orbiting body (m in this case), has been replaced by the *reduced mass*, $Mm/(M+m)$. Note that when $M >> m$, the reduced mass becomes m. However, the above expression is general and applies to general masses M and m.

Alternatively, the above expression can be written as

$$m\ddot{\boldsymbol{r}} = -G\frac{(M+m)m}{r^2}\,\boldsymbol{e}_r \ , \tag{7}$$

which is again a Kepler problem for an orbiting body of mass m, in which the gravitational parameter μ is given by $\mu = G(M+m)$.

Example ***Solution to the Two Body Problem***

There are two approaches to the solution of the two-body problem. One is a direct numerical attack on equations (1) and (2); the other is to use the analytic solution of the Kepler problem, equation(7), and having found $\boldsymbol{r}(t)$, to use the equation for the position of the center of mass, $\boldsymbol{r}_G(t)$ and equation (4) to determine $\boldsymbol{r}_m(t)$ and $\boldsymbol{r}_M(t)$. The position of the center of mass is determined by the initial conditions (position and velocity) of the bodies. Consider the motion of two bodies as shown in a). The masses of the two bodies are $M = 4$ and $m = 1$; for convenience G was set equal to 10. The initial conditions (vector components) are given as $\boldsymbol{r}_m = (1,0)$, $\dot{\boldsymbol{r}}_m = (2,3)$ and $\boldsymbol{r}_M = (-2,0)$ and $\dot{\boldsymbol{r}}_M = (-2,0)$. The motion of the two bodies with time is shown in a). From the boundary conditions, we obtain the position of the center of mass with time as $\boldsymbol{r}_G = (-7/5,0) + (-6/5,3/5)t$; this position with time is shown in b). The bodies "orbit" about the instantaneous position of the center of mass.