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Greatest Common Divisors and Least Common Multiples

Remember the goal of this unit is to build up some of the tools we need for more sophisticated number systems, like the fractional numbers. In this section we discuss the following two concepts:

Definition 1. *Let a and b be whole numbers not both 0. The largest number which is a divisor of both a and b is called the* ***greatest common divisor*** *or GCD of a and b. In symbolic shorthand, the GCD of a and b is written as* $\text{GCD}(a, b)$.

Definition 2. *Let a and b. The smallest number which is a multiple of both a and b is called the* ***least common multiple*** *or LCM of a and b. In symbolic shorthand, the LCM of a and b is written as* $\text{LCM}(a, b)$.

- The GCD of 0 and 0 is not defined since every number is a divisor of zero, so there can't possibly be a largest one.
- Note that for any non-zero whole number a, $\text{GCD}(0, a) = a$ since the largest divisor of a is a and 0 is divisible by a.
- Note that $\text{GCD}(a, b) \leq a$ and $\text{GCD}(a, b) \leq b$. (Why?)
- Note that $\text{LCM}(a, b) \leq a \cdot b$. (Why?)

Given these definitions, we now must come up with some ways to compute the GCD and LCM of specific numbers.

1. Computing the GCD

   - Method 1: Intersection of Sets.

     To find $\text{GCD}(a, b)$ write out in list notation the set of divisors of a and the set of divisors of b. The intersection of these two sets is the set of common divisors and the largest number in the intersection is the greatest common divisor. For example, if $a = 45$ and $b = 18$, then the divisors of 45 gives the set

     $$D_{45} = \{1, 3, 5, 9, 15, 45\}$$

     and

     $$D_{18} = \{1, 2, 3, 6, 9, 18\}.$$

     The set of common divisors is the set

     $$D_{45} \cap D_{18} = \{1, 3, 9\}$$

     and the largest element in this set is 9. So $\text{GCD}(45, 18) = 9$.

- Method 2: Prime Factorization Method.
  To find the GCD of two numbers a and b find the prime power factorization of a and the prime power factorization of b. The GCD is the largest possible product of prime powers that a and b have in common. For example, if $a = 45$ and $b = 18$, then we know

$$a = 3^2 \cdot 5^1, \qquad b = 2 \cdot 3^2.$$

  The only common prime in these factorizations is 3 and the largest power of three that appears in both factorizations is 3^2. So the $GCD(45, 18) = 3^2 = 9$.
  If $a = 504$ and $b = 3675$, then

$$504 = 2^3 \cdot 3^2 \cdot 7^1, \qquad 3675 = 3^1 \cdot 5^2 \cdot 7^2$$

  and the common primes are 3 and 7. The largest power of 3 which appears in both is 3^1. The largest power of 7 that appears in both is 7^1 so the GCD is $3 \cdot 7 = 21$.

2. Computing the LCM

- Method 1: Intersection of sets.
  To find LCM(a, b), we first write out in list notation the set of multiples of a and the set of multiples of b. The intersection of these two sets is the set of common multiples and the smallest number in this intersection is the LCM. For example, if $a = 9$ and $b = 15$, then

$$M_a = \{9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99, 108, 117, 126, 135, ...\}$$

$$M_b = \{15, 30, 45, 60, 75, 90, 105, 120, 135, ...\}.$$

  Then

$$M_9 \cap M_{15} = \{45, 90, 135, ...\}$$

  and the smallest element of this set is clearly 45. So the LCM$(9, 15) = 45$.

- Method 2: Prime Factorization Method.

  To find LCM(a, b), compute the prime product factorizations of a and b. The LCM is the product of prime powers in the factorization with the larger exponents.
  For example, if $a = 9, b = 15$ then

$$a = 3^2$$

$$b = 3^1 \cdot 5^1.$$

  The largest power of 3 in these factorizations is 3^2. The largest power of 5 in these factorizations is 5^1 so the LCM is $3^2 \cdot 5^1 = 45$.
  If $a = 270$ and $b = 630$ then

$$a = 2^1 \cdot 3^3 \cdot 5^1$$

and

$$b = 2^1 \cdot 3^2 \cdot 5^1 \cdot 7.$$

The largest power of 2 in these factorizations is 2^1. The largest power of 3 is 3^3. The largest power of 5 is 5^1. The largest power of 7 is 7^1. So the prime factorization is $2^1 \cdot 3^3 \cdot 5^1 \cdot 7^1$.