Math 231E. Fall 2013. Lecture 3A.

Limits using Taylor Series.

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1 Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for x near a, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} g(x) = L$.

Example 7. We compute

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x}\right).$$

We cannot use a Limit Law, since $\lim_{x\to 0} \sin(1/x)$ does not exist. However, let us Squeeze it:

$$-1 < \sin(1/x) < 1$$
$$-x^2 < x^2 \sin(1/x) < x^2,$$

and we know

$$\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0,$$

so

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

See Figure 4.

2 Computing limits using Taylor series

Example 8. Let us now consider the limit

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

We cannot use the Limit Law, since the denominator goes to zero. We know that one way to do this is l'Hôpital's Rule, but if we have Taylor series there is a better way to go.

Recall the Taylor series for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7),$$

so

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6).$$

It is easy to see if we take the limit as $x \to 0$, the right-hand side goes to one, so

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

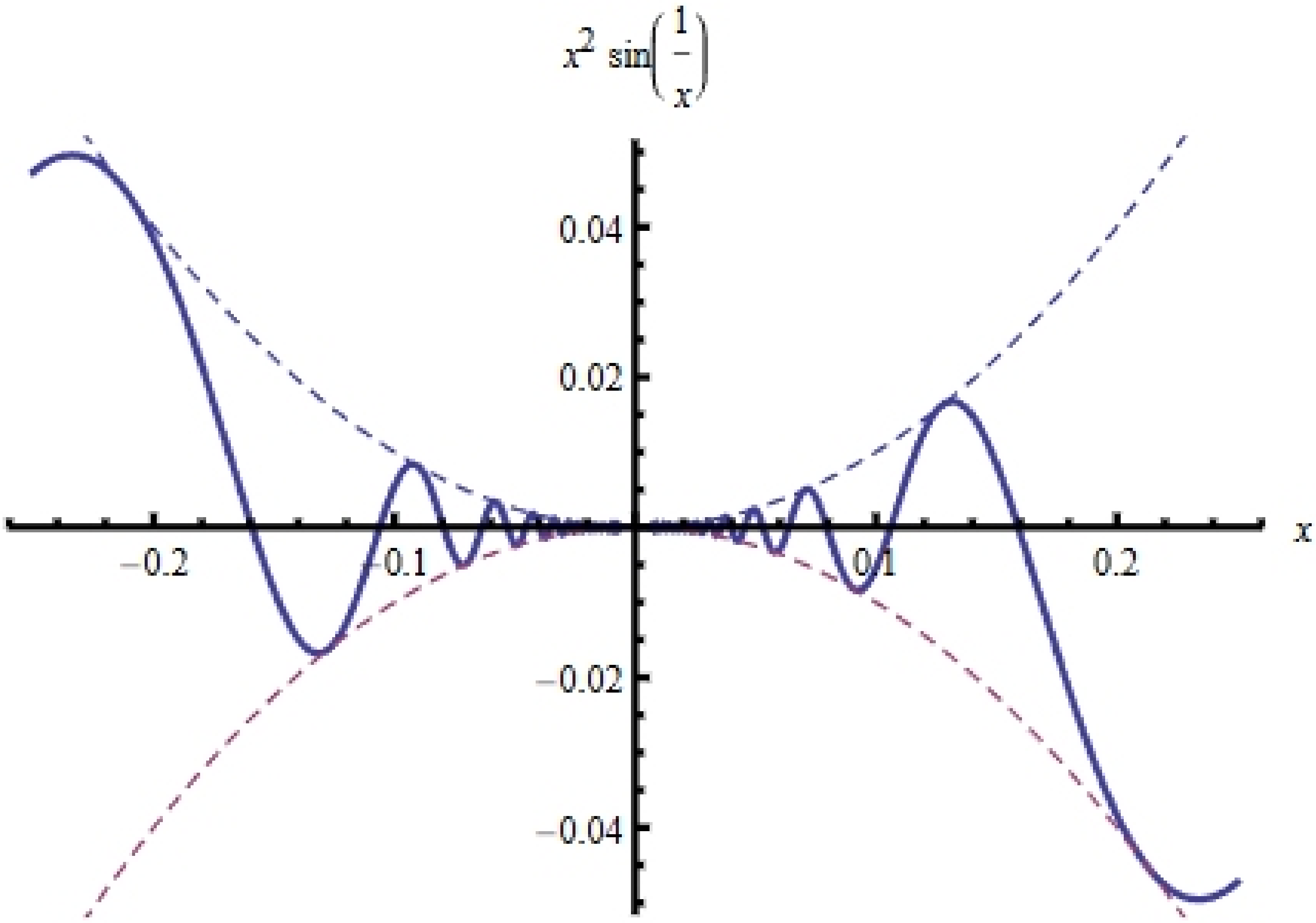

Figure 4: Plot of $x^2 \sin(1/x)$, and the envelopes $x^2, -x^2$

In fact, we can use Taylor series to derive l'Hôpital's Rule, as follows: let us say that $\lim_{x\to a} f(a) = \lim_{x\to a} g(x) = 0$. Then we compute

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a) + f'(a)(x-a) + O(x-a)^2}{g(a) + g'(a)(x-a) + O(x-a)^2}.$$

We know that $f(a) = g(a) = 0$, so

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a) + O(x-a)^2}{g'(a)(x-a) + O(x-a)^2} = \lim_{x\to a} \frac{f'(a) + O(x-a)}{g'(a) + O(x-a)},$$

where we got the last equality by dividing by $(x-a)$. But we can then use the Limit Law (as long as $g'(a) \neq 0$) and obtain

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 9. We can do much more complicated examples using Taylor series. For example, say that we want to compute

$$\lim_{x\to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)}.$$

Let us use Taylor series. We have

$$\begin{aligned}
\cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6), \\
\cos(x^2) &= 1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12}), \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4), \\
e^{x^4} &= 1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16}), \\
\sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7), \\
\sin(x^4) &= x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28}).
\end{aligned}$$

So we have

$$\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \frac{\left(1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12})\right) - \left(1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16})\right)}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}.$$

Adding like terms in the numerator gives

$$\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \frac{-\frac{3}{2}x^4 - \frac{11}{24}x^8 + O(x^{12})}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}.$$

We see that every term in the expression is divisible by x^4, so divide this out to obtain

$$\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \frac{-\frac{3}{2} - \frac{11}{24}x^4 + O(x^8)}{1 - \frac{x^8}{6} + \frac{x^{16}}{120} + O(x^{24})},$$

and taking limits as $x \to 0$ on both sides gives

$$\lim_{x \to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = -\frac{3}{2}.$$