Math 231E. Fall 2013. Lecture 3B.

Limits at Infinity.

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1 Limits at ∞ and horizontal asymptotes

We now would like to define limits as $x \to \infty$, and we can do so in the following manner:

We say that

$$\boxed{\lim_{x\to\infty} f(x) = L, \text{ if } \lim_{x\to 0+} f\left(\frac{1}{x}\right) = L.}$$

In short, we first plug in $1/x$ for x, then take the limit $x \to 0$ from the right.

Example 10. We want to compute

$$\lim_{x\to\infty} \frac{3x^2+2x-7}{x^2-4x+1}.$$

We plug in $1/x$ for x to obtain

$$\frac{3(1/x)^2+2(1/x)-7}{(1/x)^2-4(1/x)+1} = \frac{\dfrac{3}{x^2}+\dfrac{2}{x}-7}{\dfrac{1}{x^2}-\dfrac{4}{x}+1} = \frac{1/x^2}{1/x^2}\cdot\frac{3+x-7x^2}{1-4x+x^2},$$

and taking the limit $x \to 0+$, we obtain 3.

We also say that

$$\boxed{\lim_{x\to-\infty} f(x) = L, \text{ if } \lim_{x\to 0-} f\left(\frac{1}{x}\right) = L.}$$

Example 11. Let us compute

$$\lim_{x\to\infty} \frac{e^x}{x^n},$$

for some fixed $n > 0$. We know that $e^x > x^{n+1}/(n+1)!$. (Why is this?) Then we have

$$\frac{e^x}{x^n} > \frac{x^{n+1}}{x^n(n+1)!} = \frac{x}{(n+1)!}.$$

Now,

$$\lim_{x\to\infty} \frac{x}{(n+1)!} = \lim_{x\to 0+} \frac{1/x}{(n+1)!} = \frac{1}{(n+1)!}\lim_{x\to 0+}\frac{1}{x} = \infty.$$

Therefore

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty.$$

Example 12. Similarly, we would like to compute

$$\lim_{x\to\infty} x^n e^{-x}.$$

From the arguments above, we have that

$$\frac{x^n}{e^x} < \frac{(n+1)!}{x}.$$

Also, for $x > 0$, we have $x^n e^{-x} > 0$, so

$$0 < \frac{x^n}{e^x} < \frac{(n+1)!}{x}.$$

We compute

$$\lim_{x\to\infty} \frac{(n+1)!}{x} = \lim_{x\to 0+} \frac{(n+1)!}{1/x} = (n+1)! \lim_{x\to 0+} x = 0,$$

so by the Squeeze Theorem we have

$$\lim_{x\to\infty} x^n e^{-x} = 0.$$

This is why we would say that "e^{-x} decays to zero faster than **any** polynomial as $x \to \infty$".

2 Asymptotes and graphing

See Text.