Math 231E. Fall 2013. Lecture 5A.

Related rates

We would like to solve problems where multiple variables are changing, and we would like to understand the relationship between the various rates of change.

We do some examples:

Example 20. Question: Assume that water is being pumped into a balloon at $50\,\mathrm{cm}^3/\mathrm{s}$. How fast is the radius increasing when the radius is $10\,\mathrm{cm}$?

We know that the relationship between volume and radius is

$$V = \frac{4}{3}\pi r^3.$$

Differentiating both sides of this equation gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt},$$

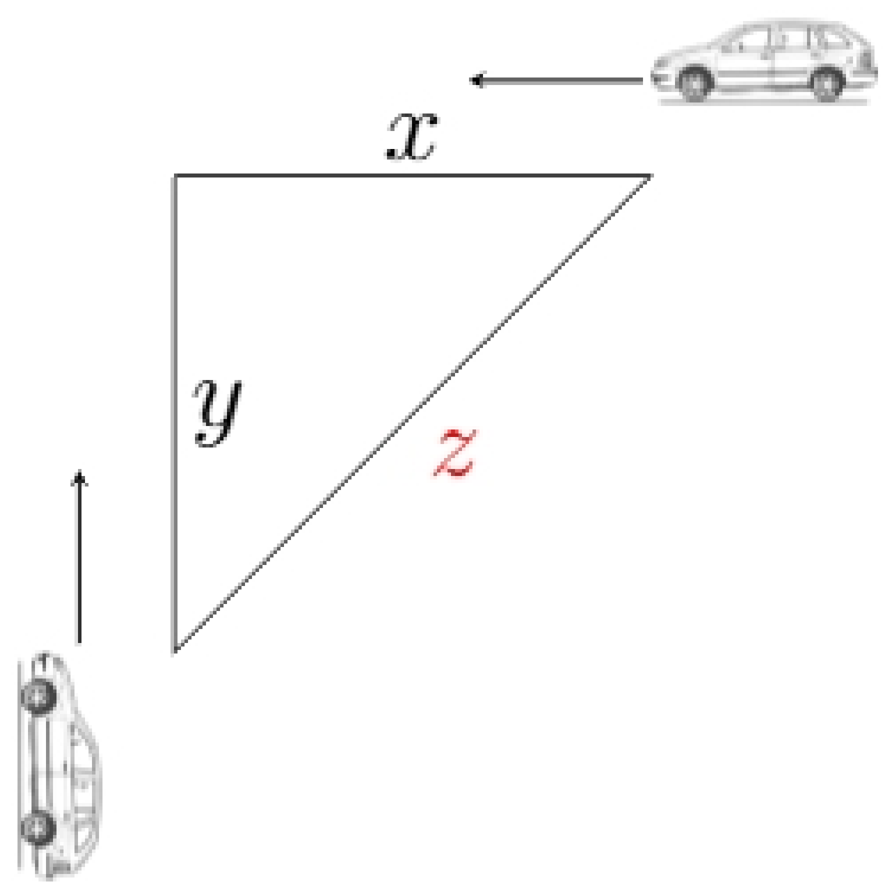
so

$$\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{50\,\mathrm{cm}^3/\mathrm{s}}{4\pi(10\,\mathrm{cm})^2} = \frac{1}{8\pi}\frac{\mathrm{cm}}{\mathrm{s}}.$$

Since $1/8\pi = .0397\cdots$, our answer is approximately $.397\,\mathrm{mm/s}$.

Example 21. Question: Two cars are approaching an intersection, one from the east and one from the south. Let's say that each car is traveling 60 mph towards the intersection, and they are currently each one mile from the intersection. How fast is the distance between the two decreasing?

We need to choose some coordinate system for these cars. Let us put the intersection at the origin, define the distance of the east-west car from the origin as x, and define the distance from the north-south car from the origin as y. We also define the distance between the cars as z.



We know from the Pythagorean Theorem that $z^2 = x^2 + y^2$. Differentiate both sides to obtain

$$\begin{aligned} 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt}, \\ z\frac{dz}{dt} &= x\frac{dx}{dt} + y\frac{dy}{dt}, \\ &= (1\,\mathrm{mi})(-60\,\mathrm{mph}) + (1\,\mathrm{mi})(-60\,\mathrm{mph}) \\ &= -120\frac{\mathrm{mi}^2}{\mathrm{hr}}. \end{aligned}$$

We also know that $z = \sqrt{2}\,\mathrm{mi}$, so

$$\frac{dz}{dt} = \frac{-120\,\mathrm{mi}^2/\mathrm{hr}}{\sqrt{2}\,\mathrm{mi}} = -\frac{120}{\sqrt{2}}\ \mathrm{mph}.$$

We can compute that $120/\sqrt{2} \approx 84.85$.