Math 231E. Fall 2013. Lecture 5C.

Max/Min

From here on out, I will stand for an interval. It can be open, closed, or semi-open.

Definition 2. The **absolute maximum** of f on I is the largest value f takes on I. Specifically, we say that M is the absolute maximum of f if there is an x with $f(x) = M$, and, for any $y \in I$, $f(y) \leq M$.

Note! it must be a value of $f(x)$. The function $f(x) = 2x + 1$ on the interval $(3, 5)$ has no absolute maximum! Also, the function $f(x) = 1/x$ on $(0, \infty)$ also has no maximum.

Definition 3. A **local maximum** is a value $f(c)$ such that $f(c) \geq f(x)$ for all x near c.

(The same definitions work for absolute and local minimum in the obvious manner.)

Theorem 9 (Extreme Value Theorem)**.** If $f(x)$ is continuous on $[a, b]$, then f attains an absolute maximum and an absolute minimum in $[a, b]$.

Theorem 10 (Fermat's (not last) Theorem)**.** If $f(x)$ has a local maximum, or local minimum, at c, and $f'(c)$ exists, then $f'(c) = 0$.

Proof. Say $f(x)$ has a local maximum at c (the proof is similar for a minimum). For $h > 0$ small enough, we have

$$f(c+h) \leq f(c).$$

This means

$$\begin{aligned} f(c+h) - f(c) &\leq 0, \\ \frac{f(c+h) - f(c)}{h} &\leq 0, \\ \lim_{h \to 0+} \frac{f(c+h) - f(c)}{h} &\leq 0. \end{aligned}$$

We also have for $h < 0$ and small enough, that $f(c+h) \leq f(c)$, or

$$\begin{aligned} f(c+h) - f(c) &\leq 0, \\ \frac{f(c+h) - f(c)}{h} &\geq 0, \\ \lim_{h \to 0-} \frac{f(c+h) - f(c)}{h} &\geq 0. \end{aligned}$$

(**Note!** Why did the inequality flip??) But the limit only exists if both the left-hand and right-hand limits are the same, so this limit must be zero. □

Definition 4. A **critical point** is a number c such that $f'(c) = 0$ or $f'(c)$ does not exist.

Algorithm to find absolute max/min

- Find critical points of f,
- Evaluate f at all of these points,
- Evaluate f at the endpoints,
- the absolute max and min must be one of these.

Let us work a few examples:

Example 24. Given a fixed perimeter, how do we maximize and minimize the area of a rectangle?

Let us consider a fixed perimeter of $100\,\mathrm{m}$. We have $2x + 2y = 100\,\mathrm{m}$, and the area is $A = xy$. Let us solve for y and get $y = 50\,\mathrm{m} - x$, so

$$A(x) = x(50\,\mathrm{m} - x) = 50\,\mathrm{m}x - x^2.$$

The domain of validity for this function is clearly $0\,\mathrm{m} \leq x \leq 50\,\mathrm{m}$, since we cannot allow x or y to be negative. We then check for critical points: $A'(x) = 50\,\mathrm{m} - 2x$, so setting this equal to zero gives $x = 25\,\mathrm{m}$. Thus we have to check three points; the critical point and the two boundary points:

$$A(0\,\mathrm{m}) = 0\,\mathrm{m}^2, \quad A(25\,\mathrm{m}) = (25\,\mathrm{m})(25\,\mathrm{m}) = 625\,\mathrm{m}^2, \quad A(50\,\mathrm{m}) = 0\,\mathrm{m}^2.$$

Clearly the interior point is an absolute maximum, and the two boundary points are absolute minima. Thus we have proved that the optimal rectangle is a square; the pessimal rectangle is the degenerate one where we choose one side length or the other to be zero.

We can generalize this to an arbitrary perimeter L. If we form a rectangle with width x and height y, then the perimeter is $L = 2x + 2y$, while the area is $A = xy$. Since L is the constraint, let us write $y = L/2 - x$, and then the area becomes

$$A(x) = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2.$$

The domain of validity of this function is $0 \leq x \leq L/2$ (if $x > L/2$ then y would be negative!). Again we find

$$A'(x) = \frac{L}{2} - 2x,$$

and thus our critical point is $x = L/4$. Again, we check the boundary points, and we obtain

$$A(0) = 0, \quad A(L/4) = (L/4)(L/4) = \frac{L^2}{16}, \quad A(L/2) = 0.$$