Math 231E. Fall 2013. Lecture 7A.

Integration by Parts

Just like u-substitution is "reverse chain rule", we have the technique of integration by parts, which is "reverse product rule".

Recall that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives

$$\int_a^b \frac{d}{dx}(f(x)g(x))\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx,$$

and the FTC gives us that

$$f(b)g(b) - f(a)g(a) = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx,$$

or, in indefinite form:

$$\int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x).$$

If we write

$$u = f(x), \quad v = g(x), \quad du = f'(x)\,dx, \quad dv = g'(x)\,dx,$$

then we write this as

$$\int u\,dv + \int v\,du = uv,$$

or

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Good news: We got rid of the integral we wanted to solve!

Bad news: And replaced it with another integral....

So, of course, this may not make things better; in fact, making a wrong choice can make it worse!

Example 31. Let us consider

$$\int xe^x\,dx.$$

We want to choose $u = x$ and $dv = e^x\,dx$. It is helpful to use such a table:

$u = x$	$v = e^x$
$du = dx$	$dv = e^x\,dx$

We then have

$$\int u\,dv = uv - \int v\,du$$
$$\int xe^x\,dx = xe^x - \int e^x\,dx$$
$$= xe^x - e^x + C.$$

Example 32. Now try

$$\int x\sin(x)\,dx.$$

We want to choose

$u = x$	$v = -\cos(x)$
$du = dx$	$dv = \sin(x)\,dx$

We then have

$$\int u\,dv = uv - \int v\,du$$
$$\int x\sin(x)\,dx = -x\cos(x) - \int(-\cos(x))\,dx$$
$$= -x\cos(x) + \sin(x) + C.$$

Example 33. A tricky example. Let us consider

$$\int \ln x\,dx.$$

It's not clear what to do here because the function doesn't really look like a product. But let us choose $u = \ln x$ and $dv = dx$, then we have

$u = \ln x$	$v = x$
$du = \dfrac{dx}{x}$	$dv = dx$

We then have

$$\int u\,dv = uv - \int v\,du$$
$$\int \ln(x)\,dx = x\ln(x) - \int dx$$
$$= x\ln(x) - x + C.$$

Example 34.

$$\int e^x \sin(x)\, dx.$$

We will choose $u = \sin(x)$ and $dv = e^x\, dx$, giving

$u = \sin(x)$	$v = e^x$
$du = \cos(x)\, dx$	$dv = e^x\, dx$

We then have

$$\int u\, dv = uv - \int v\, du$$
$$\int e^x \sin(x)\, dx = e^x \sin(x) - \int e^x \cos(x)\, dx.$$

Now we see that we have an integral that we still don't know how to do, so let us try integration by parts again!

To do

$$\int e^x \cos(x)\, dx,$$

We choose $u = \cos(x)$ and $dv = e^x\, dx$, giving

$u = \cos(x)$	$v = e^x$
$du = -\sin(x)\, dx$	$dv = e^x\, dx$

We then have

$$\int u\, dv = uv - \int v\, du$$
$$\int e^x \cos(x)\, dx = e^x \cos(x) + \int e^x \sin(x)\, dx.$$

and it unfortunately looks as if we are back where we started. However, let us put it together:

$$\begin{aligned}\int e^x \sin(x)\, dx &= e^x \sin(x) - \int e^x \cos(x)\, dx \\ &= e^x \sin(x) - \left(e^x \cos(x) + \int e^x \sin(x)\, dx\right) \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\, dx.\end{aligned}$$

We can put all of the integrals on the left-hand side to get

$$2\int e^x \sin(x)\, dx = e^x \sin(x) - e^x \cos(x)$$
$$\int e^x \sin(x)\, dx = \frac{1}{2}\left(e^x \sin(x) - e^x \cos(x)\right).$$