Math 231E. Fall 2013. Lecture 8A.

Trigonometric Integrals and Trigonometric Substitution

In this lecture we will learn how to integrate expressions involving trigonometric functions, or integrals that can be converted into same.

Example 36. Let us consider the integral

$$\int \sin^5(x)\cos^2(x)\,dx.$$

Recalling the identity

$$\sin^2(x) + \cos^2(x) = 1,$$

we can write $\sin^2(x) = 1 - \cos^2(x)$, and

$$\sin^4(x) = (\sin^2(x))^2 = (1-\cos^2(x))^2 = 1 - 2\cos^2(x) + \cos^4(x).$$

Thus we can rewrite our integral

$$\begin{aligned}\int \sin^5(x)\cos^2(x)\,dx &= \int \sin(x)\sin^4(x)\cos^2(x)\,dx\\ &= \int \sin(x)(1-2\cos^2(x)+\cos^4(x))\cos^2(x)\,dx\\ &= \int \sin(x)(\cos^2(x)-2\cos^4(x)+\cos^6(x))\,dx.\end{aligned}$$

Now, since we have a $\sin(x)$ in the integral, we can write

$$u = \cos(x), \quad du = -\sin(x)\,dx,$$

giving

$$-\int (u^2 - 2u^4 + u^6)\,du = -\frac{u^3}{3} + \frac{2}{5}u^5 - \frac{u^7}{7} + C,$$

or

$$\int \sin^5(x)\cos^2(x)\,dx = -\frac{\cos^3(x)}{3} + \frac{2\cos^5(x)}{5} - \frac{\cos^7(x)}{7} + C.$$

We can see in the scenario above that the trick has nothing to do with the power of 5 on the $\sin x$, but it uses the fact that it is odd. So, for example, if we consider any integral of the form

$$\int \sin^{2k+1}(x)\cos^m(x)\,dx,$$

where k, m are integers, then we write

$$\begin{aligned}\int \sin^{2k+1}(x)\cos^m(x)\,dx &= \int \sin(x)\sin^{2k}(x)\cos^m(x)\,dx\\ &= \int \sin(x)(1-\cos^2(x))^k\cos^m(x)\,dx,\end{aligned}$$

and then we make the substitution $u = \cos(x)$, $du = -\sin(x)\,dx$, and we obtain

$$\boxed{\int \sin^{2k+1}(x)\cos^m(x)\,dx = \int -(1-u^2)^k u^m\,du.}$$

This is a polynomial that we can always integrate. Similarly, for any integral of the form $\int \cos^{2k+1}(x)\sin^m(x)\,dx$, we can peel off all but one of the cosines:

$$\begin{aligned}\int \cos^{2k+1}(x)\sin^m(x)\,dx &= \int \cos(x)\cos^2(x)\sin^m(x)\,dx\\ &= \int \cos(x)(1-\sin^{2k}(x))^k\sin^m(x)\,dx,\end{aligned}$$

and then we make the substitution $u = \sin(x)$, $du = \cos(x)\,dx$, and we obtain

$$\boxed{\int \cos^{2k+1}(x)\sin^m(x)\,dx = \int (1-u^2)^k u^m\,du.}$$

Of course, this algorithm will only work when one (or both) of the powers on sines and cosines are odd. What about an integral of the form

$$\int \cos^2(x)\sin^4(x)\,dx?$$

Here we use the "half-angle formulas"

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x)), \quad \cos^2(x) = \frac{1}{2}(1+\cos(2x)).$$

For example, we can compute

$$\begin{aligned}\int_0^{2\pi} \sin^2(x)\,dx &= \int_0^{2\pi} \frac{1}{2}(1-\cos(2x))\,dx\\ &= \int_0^{2\pi} \frac{1}{2}\,dx - \int_0^{2\pi} \frac{1}{2}\cos(2x)\,dx\\ &= \pi - \frac{1}{2}\cos(2x)\Big|_0^{2\pi} = \pi - \frac{1}{2} + \frac{1}{2} = \pi.\end{aligned}$$

Now, how do we deal with the original problem? We can write

$$\begin{aligned}\int \cos^2(x)\sin^4(x)\,dx &= \int \frac{1}{2}(1+\cos(2x))\left(\frac{1}{2}(1-\cos(2x))\right)^2\,dx\\ &= \frac{1}{8}\int (1+\cos(2x))(1-\cos(2x))^2\,dx\\ &= \frac{1}{8}\int (1-\cos(2x)-\cos^2(2x)+\cos^3(2x))\,dx\end{aligned}$$

Now, we can deal with these four integrals separately. The constant is easy, as is the simple cosine. The cube we deal with as we described above, i.e.

$$\begin{aligned}\int \cos^3(2x) &= \int \cos^2(2x)\cos(2x)\,dx \\ &= \int (1-\sin^2(2x))\cos(2x)\,dx,\end{aligned}$$

and make the substitution $u = \sin(2x)$, $du = 2\cos(2x)\,dx$, to obtain

$$\int \cos^3(2x) = \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C.$$

We use the half-angle again to obtain

$$\int \cos^2(2x)\,dx = \int \frac{1}{2}(1+\cos(4x)) = \frac{x}{2} + \frac{1}{8}\sin(4x) + C.$$

Putting all of this together gives

$$\begin{aligned}\int \cos^2(x)\sin^4(x)\,dx &= \frac{x}{8} - \frac{1}{16}\sin(2x) - \frac{x}{16} - \frac{1}{64}\sin(4x) + \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C \\ &= \frac{x}{16} + \frac{7}{16}\sin(2x) - \frac{1}{64}\sin(4x) - \frac{1}{6}\sin^3(2x) + C.\end{aligned}$$

One of the main reasons these types of trig integrals are useful is that they allow us to deal with square roots of quadratic forms.

Example 37.

$$\int x^3\sqrt{1-x^2}\,dx.$$

If we make the substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$, then we have

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = |\cos(\theta)|.$$

As long as we take $\theta \in (-\pi/2, \pi/2)$, then $\cos(\theta) > 0$ and so $|\cos(\theta) = \cos(\theta)$. We then have

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3(\theta)\cdot\cos(\theta)\cdot\cos(\theta)\,d\theta \\ &= \int \sin^3(\theta)\cos^2(\theta)\,d\theta.\end{aligned}$$

Since we have an odd number of sin terms above, we "peel off" all but one of them and proceed as above.

Similar to the half-angle formulas, there are the "angle-addition" formulas, namely:

$$\begin{aligned}\sin A\cos B &= \frac{1}{2}(\sin(A-B)+\sin(A+B)), \\ \sin A\sin B &= \frac{1}{2}(\cos(A-B)-\cos(A+B)), \\ \cos A\cos B &= \frac{1}{2}(\cos(A-B)+\cos(A+B)).\end{aligned}$$