Math 231E. Fall 2013. Lecture 8B.

Partial Fractions, Strategy for Integration

We know how to do integrals of the form

$$\int \frac{dx}{x-3} = \ln|x-3| + C,$$

and we know how to do integrals of the form

$$\int \frac{dx}{x^2+1} = \arctan(x) + C.$$

We can also do integrals of the form

$$\int \frac{3x\,dx}{x^2+1} = \int \frac{\frac{3}{2}\,du}{u} = \frac{3}{2}\ln|u| + C = \frac{3}{2}|x^2+1| + C.$$

All of these integrals are of what are called rational functions:

Definition 7. A **rational function** is a function

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials.

The idea behind the technique of *partial fractions* is that it allows us to compute the antiderivative of any rational function, after some work. The main idea behind it is that we write any rational function as a sum of simpler rational functions that we know how to integrate, and more or less those three examples above are the prototypical examples that we need to consider.

Said another way, partial fractions is mostly an algebraic technique that allows us to rewrite integrands in a more useful form.

We will first consider a couple of examples.

Example 39. Let us consider the integral

$$\int \frac{dx}{(x-1)(x-2)}.$$

Let us guess that we can write

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

for some numbers A, B. If we can do so, then we will be able to do these integrals. Let us simplify that previous expression by putting everything over a common denominator

$$\frac{1}{(x-1)(x-2)} = \frac{A(x-2)}{(x-1)(x-2)} + \frac{B(x-1)}{(x-1)(x-2)},$$

and then equating numerators

$$1 = A(x-2) + B(x-1) = (A+B)x + (-2A-B).$$

(We could also just have multiplied out the denominators, and obtained the same thing.)

There are then two ways to solve for A, B. First, if we consider the equation

$$(A+B)x + (-2A-B) = 1,$$

and note that it must be true for all x, this gives the system

$$A+B=0, \quad -2A-B=1,$$

which has solution $A=-1, B=1$. The second way is to consider the equation

$$1 = A(x-2) + B(x-1),$$

and note that it simplifies considerably for two special cases of x, namely $x = 1, 2$. If we plug in $x = 1$ we obtain

$$1 = A(-1) = -A,$$

so $A = -1$, and if we plug in $x = 2$ we obtain

$$1 = B(1) = B,$$

so $B = 1$.

Either way, we have found that

$$\frac{1}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{1}{x-2}.$$

Thus

$$\int \frac{dx}{(x-1)(x-2)} = -\ln|x-1| + \ln|x-2| + C = \ln\left|\frac{x-2}{x-1}\right| + C.$$