Math 231E. Fall 2013. Lecture 8C.

Partial Fractions II

Example 40. We can actually generalize this example considerably. Let us now consider the integral

$$\int \frac{(\alpha x + \beta)\, dx}{(x-1)(x-2)},$$

where α, β are any numbers. We can then try the same decomposition:

$$\frac{(\alpha x + \beta)}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2},$$

which becomes

$$\alpha x + \beta = (A+B)x + (-2A - B).$$

This gives the two equations

$$A + B = \alpha, \quad -2A - B = \beta.$$

Adding the two equations gives

$$-A = \alpha + \beta, \text{ or, } A = -\alpha - \beta,$$

and then we have $B = \alpha - A = 2\alpha + \beta$. Thus we have

$$\frac{(\alpha x + \beta)}{(x-1)(x-2)} = \frac{-\alpha - \beta}{x-1} + \frac{2\alpha + \beta}{x-2},$$

and thus

$$\int \frac{(\alpha x + \beta)\, dx}{(x-1)(x-2)} = -(\alpha + \beta)\ln|x-1| + (2\alpha + \beta)\ln|x-2| + C.$$

This technique will always work as long as the degree of the numerator is less than the degree of the denominator, i.e. as long as it is a proper fraction. What if it is not?

Example 41. We now consider

$$\int \frac{x^3 + 4\, dx}{x^2 - 3x + 2}.$$

We first have to do long division to make this a proper fraction. We saw how to write long division as a division problem earlier in the course, so let us now repeat this technique, but in a different way.

Let us write $P(x) = x^3 + 4$ and $Q(x) = x^2 - 3x + 2$. We see then that to match the leading order terms, we have to multiply $Q(x)$ by x, so let us see what happens:

$$P(x) - xQ(x) = x^3 + 4 - x(x^2 - 3x + 2) = x^3 + 4 - x^3 + 3x^2 - 2x = 3x^2 - 2x + 4.$$

To remove the now leading order term, we have to multiply by 3, so we have

$$P(x) - xQ(x) - 3Q(x) = 3x^2 - 2x + 4 - 3(x^2 - 3x + 2) = 3x^2 - 2x + 4 - 3x^2 + 9x - 6 = 7x - 2.$$

Thus we have shown that

$$P(x) - (x+3)Q(x) = 7x - 2.$$

Now that the remainder is of a lower degree that $Q(x)$, we cannot proceed any further and we stop here. Thus we have

$$\frac{P(x)}{Q(x)} - (x+3) = \frac{7x-2}{Q(x)},$$

or

$$\frac{x^3+4\,dx}{x^2-3x+2} = x+3+\frac{7x-2}{x^2-3x+2}.$$

Of course, we know how to integrate $x+3$, and we have already done the quotient in the general case above, with $\alpha = 7$ and $\beta = -2$, so we have

$$\frac{x^3+4\,dx}{x^2-3x+2} = \frac{x^2}{2} + 3x - 5\ln|x-1| + 12\ln|x-2| + C.$$

Example 42. This technique of partial fractions can extend to multiple terms in the denominator. For example, if we want to integrate a function of the form

$$\int \frac{2x+3\,dx}{(x-1)(x-2)(x-3)(x-4)},$$

then we first write

$$\frac{2x+3}{(x-1)(x-2)(x-3)(x-4)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{D}{x-4}.$$

Expanding this out gives

$$2x+3 = A(x-2)(x-3)(x-4)+B(x-1)(x-3)(x-4)+C(x-1)(x-2)(x-4)+D(x-1)(x-2)(x-3).$$

Now, we could expand the right-hand side out and equate coefficients, but this is a case where plugging in is more efficient. If we plug in the four numbers $x = 1,2,3,4$ into this equation, we obtain the system

$$\begin{aligned}
5 &= A(-1)(-2)(-3) = -6A,\\
7 &= B(1)(-1)(-2) = 2B,\\
9 &= C(2)(1)(-1) = -2C,\\
11 &= D(3)(2)(1) = 6D.
\end{aligned}$$

This gives us

$$A = -\frac{5}{6}, \quad B = \frac{7}{2}, \quad C = -\frac{9}{2}, \quad D = \frac{11}{6}.$$

Putting this all together gives

$$\int \frac{2x+3\,dx}{(x-1)(x-2)(x-3)(x-4)} = -\frac{5}{6}\ln|x-1| + \frac{7}{2}\ln|x-2| - \frac{9}{2}\ln|x-3| + \frac{11}{6}\ln|x-4| + C.$$

Another case is if we have a repeated root in the denominator, along the lines of the following example:

Example 43. Let us consider

$$\int \frac{(2x+1)\,dx}{(x-1)^2(x-2)}.$$

It turns out for the partial fractions we need to choose three terms (since there are three factors down below), and we do this by choosing as follows:

$$\frac{2x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}.$$

Multiplying out gives

$$2x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2.$$

The trick we used before of plugging in special values will not quite work here (although a refinement of this will work later). Let us multiply out the right-hand side and obtain

$$2x+1 = A(x^2-3x+2)+B(x-2)+C(x^2-2x+1) = (A+C)x^2+(-3A+B-2C)x+(2A-2B+C),$$

which gives us the system

$$\begin{aligned} A + C &= 0, \\ -3A + B - 2C &= 2, \\ 2A - 2B + C &= 1. \end{aligned}$$

The first equation tells us that $C = -A$, so this simplifies the last two equations to

$$-A + B = 2, \quad A - 2B = 1,$$

which gives

$$A = -5, \quad B = -3, \quad C = 5,$$

and thus

$$\frac{2x+1}{(x-1)^2(x-2)} = \frac{-5}{x-1} + \frac{-3}{(x-1)^2} + \frac{5}{x-2}.$$

Thus

$$\int \frac{2x+1}{(x-1)^2(x-2)} = -5\ln|x-1| + \frac{3}{x-1} + 5\ln|x-2| + C.$$

In general, whenever we have $(x-a)$ to a power larger than one in the denominator, say $(x-a)^p$, then we must have the p terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_p}{(x-a)^p}$$

in the partial fraction expansion.

Example 44. Finally, if we ever have terms in the denominator that are quadratics with no real roots, then what we do is not factor the quadratic. However, since we will have two powers of x in the denominator, we need two parameters in the numerator, so we always choose as follows