## Series.

When we say $\pi = 3.1415926535\cdots$

what does that mean?

$$\pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \cdots$$

We mean an infinite sum.

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**Def.** Given a ~~series~~ sequence with terms $a_n$, let us define the $n^{th}$ partial sum

$$S_n = \sum_{i=1}^{n} a_i.$$

If $\lim_{n\to\infty} S_n$ exists, then we say $\sum_{i=1}^{\infty} a_i$ converges, and is equal to this limit.

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For $\pi$, we have

$a_1 = 3,\ a_2 = \frac{1}{10},\ a_3 = \frac{4}{100},\ a_4 = \frac{1}{1000}, \ldots$

$S_1 = 3,\ S_2 = 3.1,\ S_3 = 3.14,\ S_4 = 3.141, \ldots$

$$a_n = \frac{(n-1)^{st} \text{ digit after decimal}}{10^n}.$$

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**Def.** A geometric series with ratio $r$ and initial term $a$, is the sum

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

$$= \sum_{k=1}^{\infty} ar^k.$$

Q: Does this converge??

$S_n = a + ar + ar^2 + \cdots + ar^n$
$rS_n = \quad ar + ar^2 + \cdots + ar^n + ar^{n+1}$

$S_n - rS_n = a - ar^{n+1}$
$(1-r) S_n = a - ar^{n+1}$

$$S_n = \frac{a - ar^{n+1}}{1-r} = \frac{a(1-r^{n+1})}{1-r} \qquad (r \neq 1)$$

Q: Does limit $\lim_{n\to\infty} S_n$ exist?

$$\lim_{n\to\infty} \frac{a}{1-r}(1-r^{n+1}) = \frac{a}{1-r} \lim_{n\to\infty} (1-r^{n+1})$$

Recall: if $|r|<1$, $\lim_{n\to\infty} r^{n+1} = 0$
if $|r|>1$, $r=-1$, lim DNE.

Finally, if $r=1$, $S_n = na$, $\lim_{n\to\infty} na$ diverges.

Theorem: The geometric series $\sum_{k=1}^{\infty} ar^k$ converges if and only if $|r|<1$.

If so,
$$\sum_{k=1}^{\infty} ar^k = \frac{a}{1-r}.$$

Eg.

Consider. $3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} - \frac{32}{81}$

$$a_n = a r^n, \quad a = 3, \; r = \left(-\frac{2}{3}\right).$$

$$\sum_{n=1}^{\infty} 3\left(-\frac{2}{3}\right)^n \qquad \left|-\frac{2}{3}\right| < 1 \; \checkmark$$

$$= \frac{3}{1-\left(-2/3\right)} = \frac{3}{5/3} = \boxed{\frac{9}{5}}.$$

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$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}. \quad ?$$

Note, we can write $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$$

$$= 1 \cancel{- \frac{1}{2}} \cancel{+ \frac{1}{2}} \cancel{- \frac{1}{3}} \cancel{+ \frac{1}{3}} \cancel{- \frac{1}{4}} \cancel{+ \frac{1}{4}} - \frac{1}{5} + -$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S_n = 1.$$