Recall: Test For Divergence

T. If $\lim_{n\to\infty} a_n$ D.N.E., or $\to L \neq 0$, then

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Several New Tests.

1) Integral Test. Suppose $f(x)$ is continuous, positive, decreasing function on $[1,\infty)$.
Let $a_n = f(n)$. Then
$\sum a_n$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.
In other words, both converge or both diverge

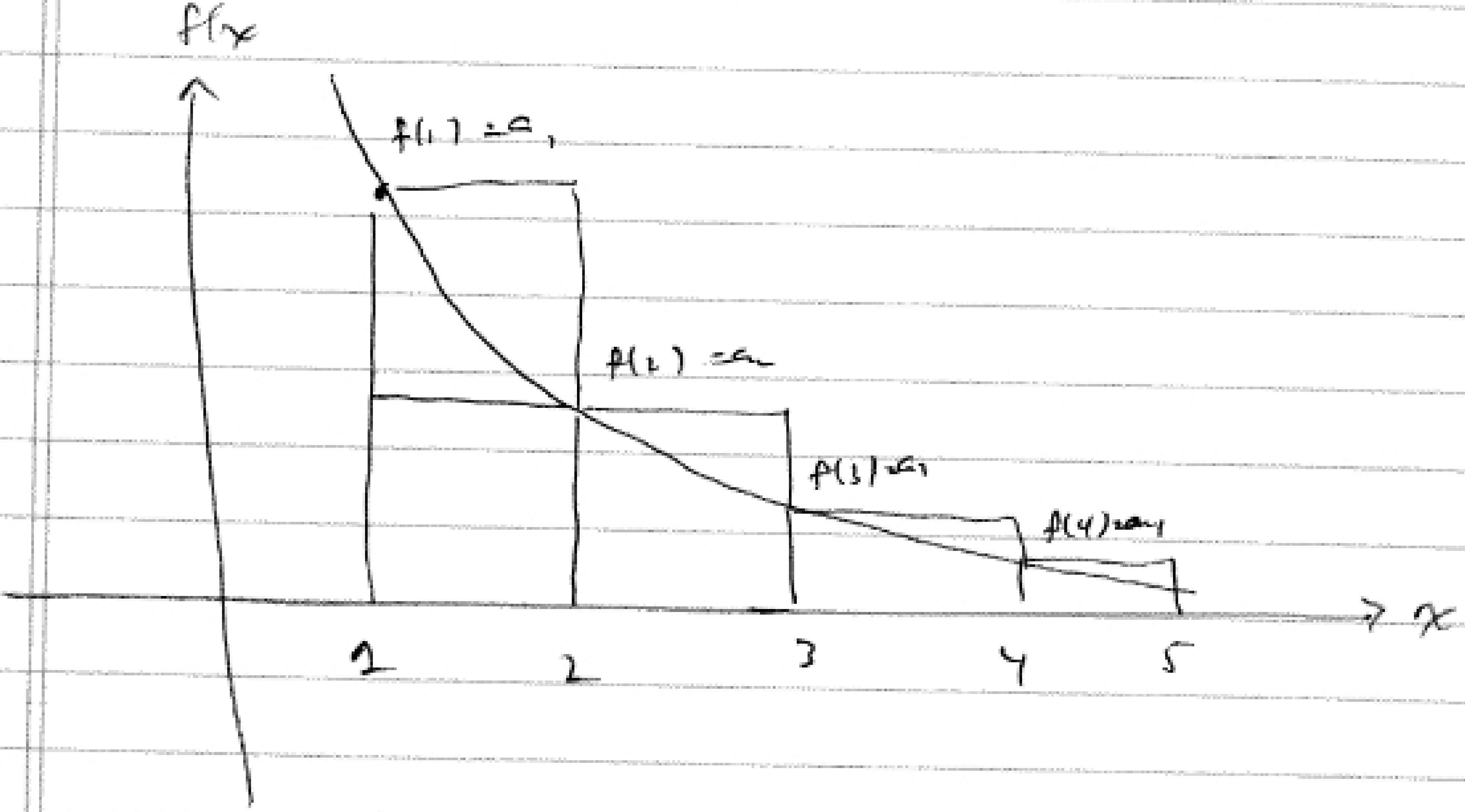
2) Comparison Test. Suppose $a_n, b_n \geq 0$. , $a_n \leq b_n$.
2a) If $\sum a_n$ diverges, $\sum b_n$ diverges
2b) If $\sum b_n$ converges, $\sum a_n$ converges.

3) Limit Comparison Test. Assume $a_n, b_n \geq 0$. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad \text{where } 0 < c < \infty, \text{ then}$$

either both sums converge, or both sums diverge.

1) Integral Test.



$$a_2 \leq \int_1^2 f(x)\,dx \leq a_1$$

$$a_3 \leq \int_2^3 f(x)\,dx \leq a_2$$

$$a_4 \leq \int_3^4 f(x)\,dx \leq a_3$$

$$\vdots$$

$+$

$$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n$$

E.g.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges if } p>1$$
$$\text{diverges if } p\leq 1.$$

why?

$$\int_1^{\infty} \frac{dx}{x^p} \quad \text{conv if } p>1$$
$$\text{diverges if } p\leq 1$$

However!!! $\int_1^{\infty} f(x)dx \neq \sum_{n=1}^{\infty} a_n$ in general

$$\int_1^{\infty} \frac{dx}{x^2} = \frac{-1}{x}\Big]_{x=1}^{x=\infty} = 0-(-1)=1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad \boxed{\frac{\pi^2}{6} \neq 2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450} \cdots\cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = ?$$ Apéry's Constant

Theorem: (Apéry, 1978) $\zeta(3)$ is irrational.

Fact: $\zeta(5)$ = not even known if it is rational?