11.5 Alternating Series Test.

Def. An alternating series is a series where the terms are alternately positive & negative, i.e. the signs are alternating.

e.g. $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

We continue the convention that $a_n = n^{th}$ # in seq.
we write $b_n = |a_n|$.
Then an alternating series is of the form

$$a_n = (-1)^n b_n \quad \text{OR} \quad a_n = (-1)^{n+1} b_n.$$

A.S.T. Let $a_n = (-1)^{n+1} b_n$, and consider the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$$

If
1) $b_{n+1} \leq b_n$ for all n (non-increasing)
2) $\lim_{n\to\infty} b_n = 0$, then

series converges.

(All the same if we write $a_n = (-1)^n b_n$ as well)
Also, all the same if we change #1 to
$b_{n+1} \leq b_n$ for n sufficiently large.

E.g. Alternating Harmonic Series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

This converges. why? $\quad b_n = \frac{1}{n} \quad b_{n+1} < b_n$ ✓

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0 \quad ✓$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 6n}{7n+2} \qquad b_n = \frac{6n}{7n+1}$$

$$\lim_{n\to\infty} b_n = \frac{6}{7} \neq 0 \quad \times$$

cannot use A.S.T.

however, $\lim_{n\to\infty} a_n$ D.N.E $\Rightarrow \sum a_n$ diverges by D.T.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^3+3} \qquad \lim_{n\to\infty} b_n = 0 \quad ✓$$

is b_n decreasing??

$$f(x) = \frac{x^2}{2x^3+3}. \qquad f'(x) = \frac{2x(2x^3+3) - x^3(6x^2)}{(2x^3+3)^2}$$

$$= \frac{6x + 4x^4 - 6x^5}{(2x^3+3)^2}$$

Clearly, this becomes negative for x large enough.

$\Rightarrow b_{n+1} \leq b_n$ for n large enough. ✓

Estimating Alternating Sum Error.

Say $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges, call sum S.

say $\lim_{n\to\infty} b_n = 0$, $b_{n+1} \le b_n$ for n large enough.

Define $R_N = S - S_N$

$$= \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

$$= \sum_{n=N+1}^{\infty} a_n .$$

Then

$$|R_n| \le b_{n+1} .$$

E.g. $\sum_{n=0}^{\infty} (-1)^n / n!$. Show converges, estimate to 2 decimal places.

Check. A.S.T. $b_n = \frac{1}{n!}$ $\lim_{n\to\infty} b_n = 0$ ✓

$b_{n+1} = \frac{1}{(n+1)!} = \frac{1}{n+1} \frac{1}{n!}$

$= \frac{1}{n+1} \cdot b_n$.

$b_{n+1} \le b_n$ ✓

Converges!

Moreover: $|R_n| \le b_{n+1}$.

$$S = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}$$

$$1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} + \cdots$$

notice: correct to 2 decimal places sometimes means "within

$\frac{1}{2} \times \frac{1}{100} = 0.005 . = \frac{1}{200}$.