Absolute Convergence + Ratio, Root Tests.

Def. A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent.

E.g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6}$ is A.C., since $\sum_{n=1}^{\infty} \frac{1}{n^6}$ is C

E.g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is not A.C., since $\sum_{n=1}^{\infty} \frac{1}{n}$ is D

Def. If a series is convergent, but not absolutely convergent, we say it is conditionally convergent.

$\rightarrow \sum (-1)^n/n$ is CC.

Theorem: AC $\Rightarrow$ C.

Pf $0 \le a_n + |a_n| \le 2|a_n|$.

If $\sum |a_n|$ is convergent, so is $\sum 2|a_n|$, so is $\sum (a_n + |a_n|)$ by comparison test.

Then $\sum a_n = \underbrace{\sum (a_n + |a_n|)}_{C} - \underbrace{\sum |a_n|}_{C} \rightarrow$ convergent.

QED.

E.g. $\sum \frac{\sin n}{n^2}$. could be ≥ 0, ≤ 0, cannot use comparison.

$$\left|\frac{\sin n}{n^2}\right| = \frac{|\sin n|}{n^2} \le \frac{1}{n^2}.$$

$$\sum \frac{1}{n^2} \text{ C} \Rightarrow \sum \left|\frac{\sin n}{n^2}\right| \text{ C} \Rightarrow \sum \frac{\sin n}{n^2} \text{ AC}$$

$$\Rightarrow \sum \frac{\sin n}{n^2} \text{ C}.$$

Moral: Minus Sign CAN ONLY HELP.

General Comparison Principle

If a_n, b_n are seq. s.t. $b_n \geq 0$, $|a_n| \leq b_n$, and $\sum b_n$ converges, then so does $\sum a_n$.

Pf. $\sum b_n$ C $\Rightarrow$ $\sum |a_n|$ C $\Rightarrow$ $\sum a_n$ AC

$\Rightarrow$ $\sum a_n$ C. QED.

Ratio Test.

Let $\sum a_n$ be a series such that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

Then:

1) If $L < 1$, $\sum a_n$ is AC (and thus C)

2) If $L > 1$, $\sum a_n$ is D.

3) If $L = 1$, no information!

Pf (Sketch)

Case 1: a_n will be dominated by geometric series with ratio $L < 1$.
$\Rightarrow$ Converge!

Case 2: a_n will dominate a geometric series with ratio $L > 1$,
$\Rightarrow$ diverge.

QED

E.g. $L = 1$ really means NO information.

2) $\sum \frac{1}{n^2}$. $a_n = \frac{1}{n^2}$. $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n^2}{(n+1)^2}$

$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$.

but series converges.

2) $\sum \frac{1}{n}$. $a_n = \frac{1}{n}$. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1$, but series diverges.

E.g. $\sum_{n=1}^{\infty} \frac{(-1)^n n^{17}}{3^n}$. $a_n = \frac{(-1)^n n^{17}}{3^n}$, $a_{n+1} = \frac{(-1)^{n+1}(n+1)}{3^{n+1}}$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{17}}{n^{17}} \frac{3^n}{3^{n+1}} =$

$= \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)^{17}}{n^{17}} = \frac{1}{3} \lim_{n\to\infty} \frac{(1+\frac{1}{n})^{17}}{1} = \frac{1}{3} \cdot 1 = \frac{1}{3}$.

since $\frac{1}{3} < 1$, AC.

$\sum_{n=1}^{\infty} \frac{2^n}{n^{6000}}$. $a_n = \frac{2^n}{n^{6000}}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)^{6000}}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{2^n} \frac{(n+1)^{6000}}{n^{6000}} = 2 \cdot \frac{(1+\frac{1}{n})^{6000}}{1}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2$. $2 > 1$, D.

* All about the exponential growth!

E.g. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. $a_n = \frac{n!}{n^n}$. $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$.

$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{\cancel{(n+1)} \cdot n!}{\cancel{(n+1)}(n+1)^n} = \frac{n!}{(n+1)^n}$

$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)^n} \cdot \frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$

as $n\to\infty$, exponent $\to \infty$, but abscissa $\to 1$. Indeterminate!