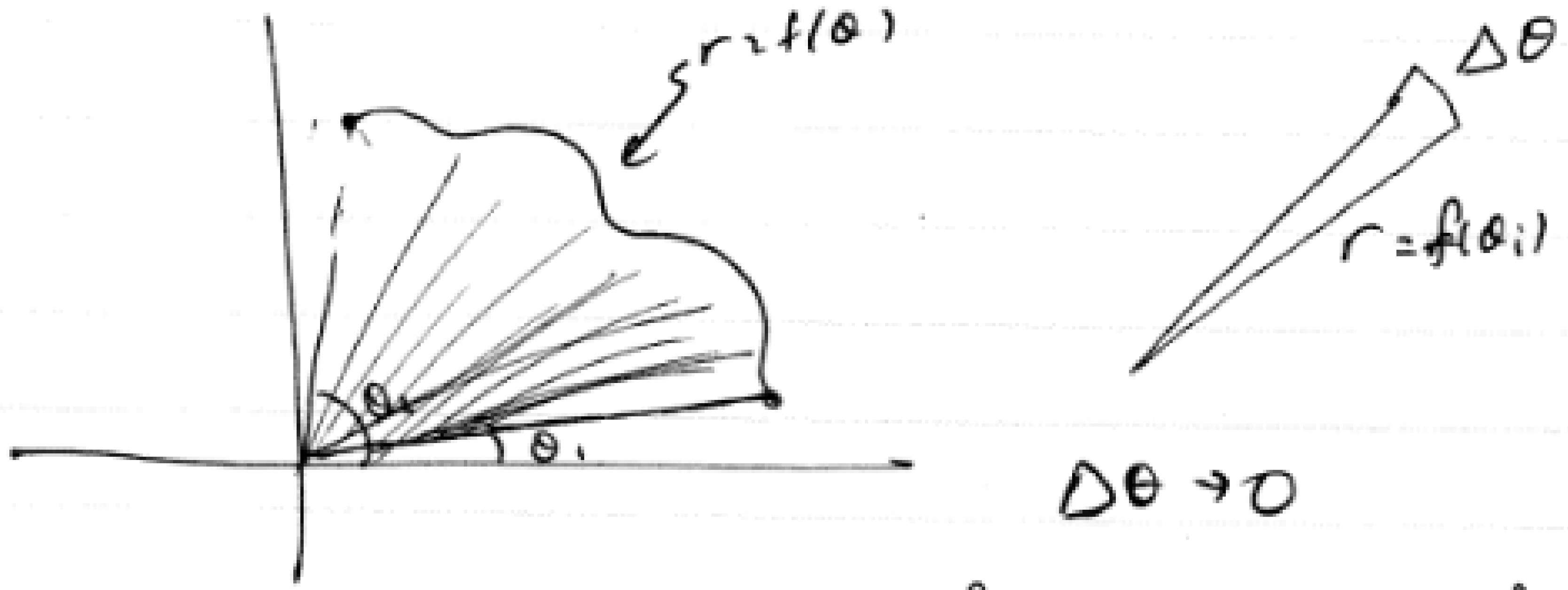
Area, Length in Polar Coordinates

area of sector of a circle:

$$A = \frac{1}{2}\theta r^2 = \frac{r^2}{2}\cdot\theta.$$

What about a more general region?

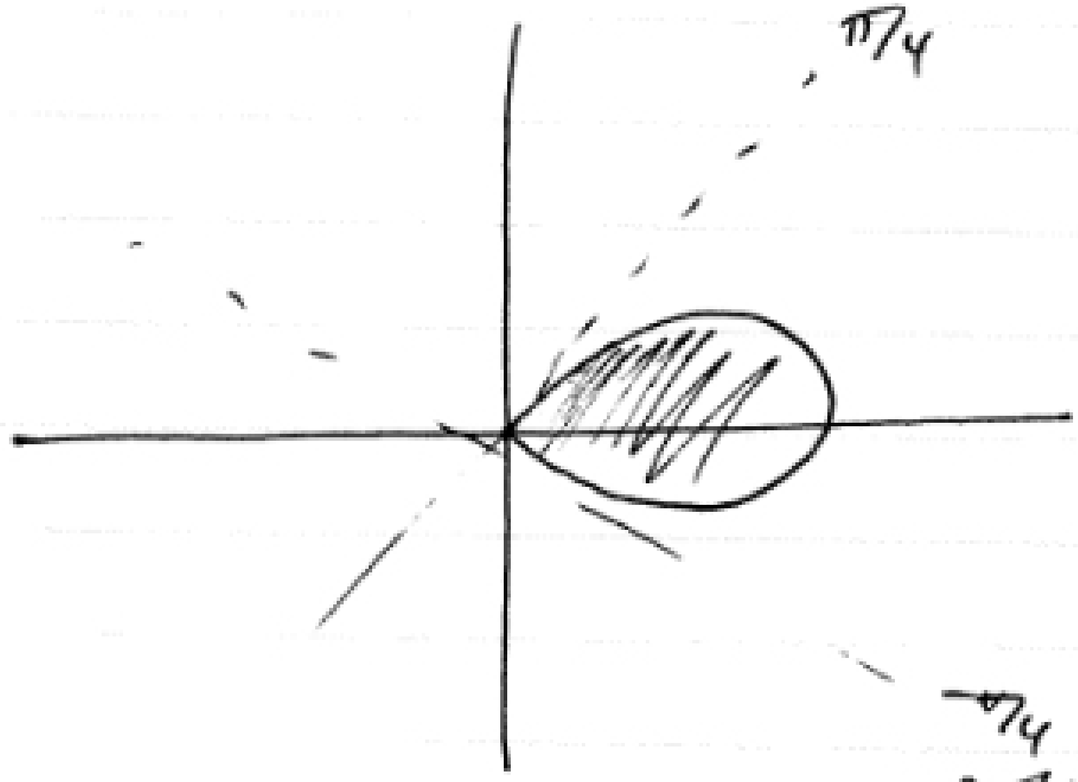


$$\frac{r^2}{2}\Delta\theta = \frac{(f(\theta_i))^2}{2}\Delta\theta$$

$\Delta\theta \to 0$

$$\int_{\theta_1}^{\theta_2} \frac{r^2}{2}\,d\theta = \int_{\theta_1}^{\theta_2} \frac{1}{2}(f(\theta))^2\,d\theta.$$

Recall: $r = \cos 2\theta$. Area of one leaf.



$$\int_{-\pi/4}^{\pi/4} \frac{(\cos(2\theta))^2}{2}\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{\cos^2(2\theta)}{2}\,d\theta$$

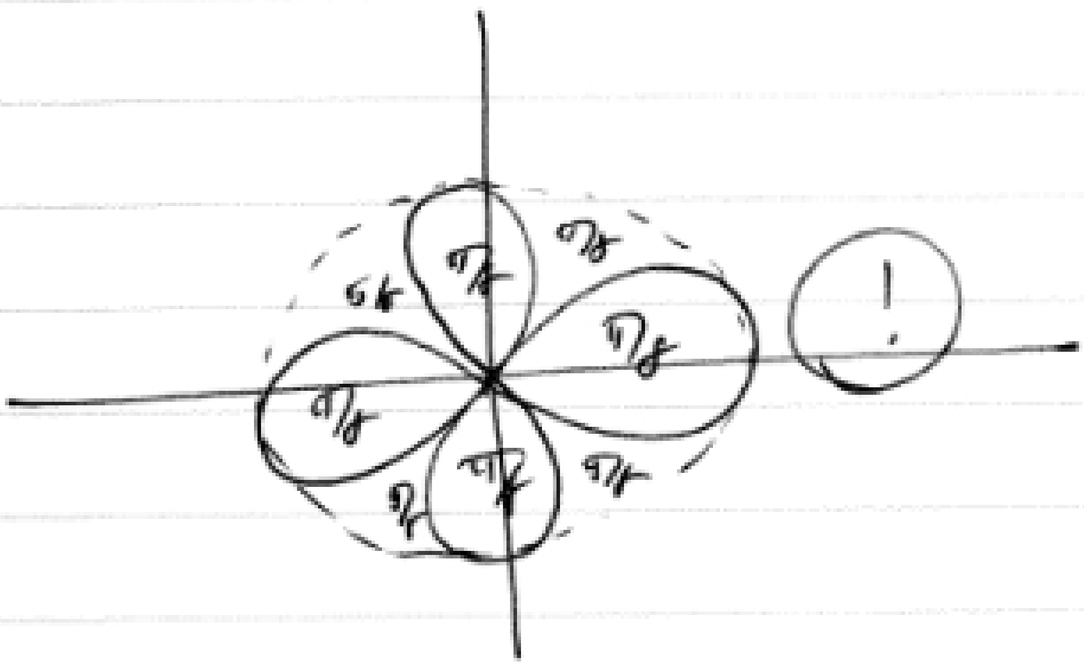
$$= \int_0^{\pi/4} \cos^2(2\theta)\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}(1+\cos(4\theta))\,d\theta$$

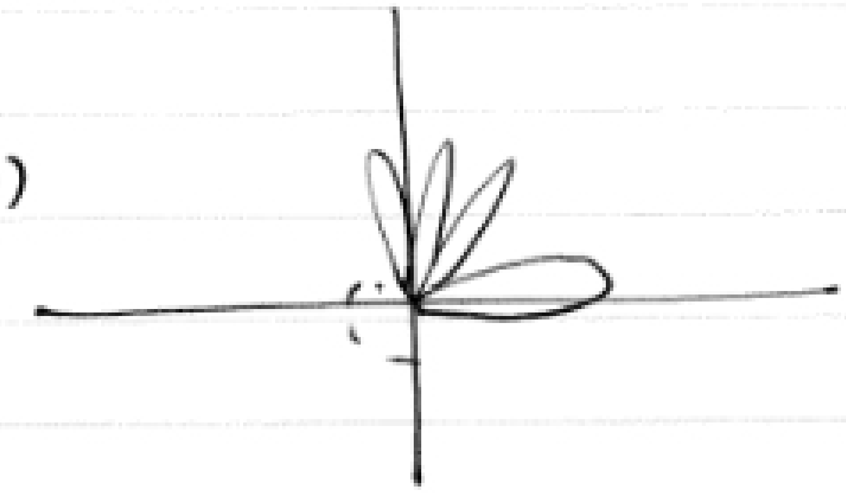
$$\frac{\theta}{2} + \frac{\sin(4\theta)}{8}\Bigg]_0^{\pi/4}$$

$$= \frac{\pi}{8} - 0 + \frac{\sin(\pi)}{8} - \frac{\sin(0)}{8} = \pi/8.$$

Whole area: $4\cdot\pi/8 = \pi/2$.



what about one leaf of $\cos(17\theta)$



lowest θ s.t. $\cos(17\theta)=0$?

$$17\theta = \pi/2$$
$$\theta = \pi/34$$

$$\frac{1}{2}\int_{-\pi/34}^{\pi/34} \cos^2(17\theta) = \int_0^{\pi/34} \cos^2(17\theta)\,d\theta$$

$$= \int_0^{\pi/34} \frac{1}{2}\left(1+\cos(34\theta)\right)d\theta = \frac{\theta}{2} + \frac{\sin(34\theta)}{68}\Bigg]_{\theta=0}^{\theta=\pi/34}$$

$$= \frac{\pi}{68} - 0 + 0 - 0 = \pi/68.$$

There are 17 leaves. $17\,\pi/68 = \pi/4$.

Arclength?. The formula for parametric A.L. if ~~x=f(θ)~~, ~~y=g(θ)~~
x, y depend on θ

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta.$$

$r = f(\theta)$

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

$\frac{dx}{d\theta} = \left(\frac{dr}{d\theta}\right)\cos\theta - r\sin\theta$

$\frac{dy}{d\theta} = \left(\frac{dr}{d\theta}\right)\sin\theta + r\cos\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2\cos^2\theta \cancel{- 2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\sin^2\theta$$
$$+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta \cancel{+ 2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2.$$

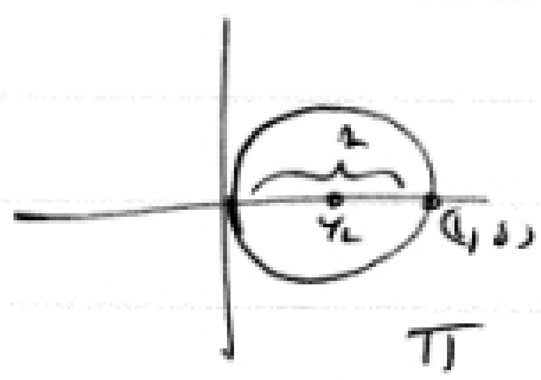
$$\underline{ds} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.$$

Length of 4-leaf clover? $r = \cos 2\theta$

$\frac{dr}{d\theta} = -2\sin 2\theta$

$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \cos^2 2\theta + 4\sin^2 2\theta$

$$\int_0^{2\pi} \sqrt{\cos^2(2\theta) + 4\sin^2(2\theta)}\, d\theta.$$



$\underline{\pi}$

$r = \cos\theta$

$\frac{dr}{d\theta} = -\sin\theta$

$$\int_0^{2\pi} \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$$
$$= \int_0^{2\pi} 1\, d\theta = \underline{2\pi}.$$

<u>WAT</u>

<u>Recall</u>: we rotated around the circle twice.