

AMS 572 Lecture Notes #2

Sept. 13, 2010.

Chapter 7: Inference on one population mean □

1 Motivation & simple random sample

Eg) We wish to estimate the average height of adult US males

→ Take a random sample.

- “Simple” random sample: every subject in the population has the same chance to be selected.

2 Introduction to statistical inference on one population mean

For a “random sample” of size n: X_1, X_2, \dots, X_n

<i> Point estimator \bar{X} → sample mean ($= \frac{X_1 + X_2 + \dots + X_n}{n} = \frac{\sum_{i=1}^n X_i}{n}$)

Other estimators: median, mode, trimmed mean, ...

<ii> Confidence Interval (C.I.)

Eg) 95% C.I. for μ

99.9999% C.I. (‘6-9’ in the manufacture industry)

<iii> Hypothesis Test

Eg) $H_0 : \mu = \mu_0$
 $H_1 : \mu \neq \mu_0$

Point Estimator, C.I., Test ⇒ Statistical Inference

- Draw some conclusion on the population (parameters of interest) based on a random sample.

3 Normal Distribution

<i> Probability Density Function (p.d.f.)

$$X \sim N(\mu, \sigma^2)$$

(X follows normal distribution of mean μ and variance σ^2)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, x \in \mathbb{R}$$

$$P(a \leq X \leq b) = \int_a^b f(x) dx = \text{area under the pdf curve bounded by a and b}$$

<ii> Cumulative Density Function (c.d.f.)

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

$$f(x) = [F(x)]' = \frac{d}{dx} F(x)$$

<iii> Z-score

Standard Normal Distribution

$$Z \sim N(0,1)$$

$$X \sim N(\mu, \sigma^2)$$

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

$$\begin{aligned} P(a \leq X \leq b) &= P\left(\frac{a - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma}\right) \\ &= P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) \end{aligned}$$

Proof) 1> Use c.d.f.

$$F_z(a) = P(Z \leq a) = P\left(\frac{X - \mu}{\sigma} \leq a\right) = P(X \leq \sigma \cdot a + \mu) = F_x(\sigma \cdot a + \mu)$$

$$f_z(a) = \frac{d}{da} F_z(a) = \frac{d}{da} F_x(\sigma \cdot a + \mu) = f_x(\sigma \cdot a + \mu) \cdot \sigma$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\sigma \cdot z + \mu - \mu)^2}{2\sigma^2}} \cdot \sigma = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \rightarrow \text{the pdf for } N(0,1)$$

2> Use p.d.f.

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$x = \sigma \cdot z + \mu$$

$$|J| = \left| \frac{dx}{dz} \right| = \left| \frac{d}{dz}(\sigma \cdot z + \mu) \right| = |\sigma| = \sigma$$

$$f_z(z) = |J| \cdot f_x(x) = \sigma \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sigma \cdot z + \mu - \mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\therefore Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

3> Use m.g.f.

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx, \text{ if } x \text{ is continuous}$$

$$= \sum_{x's} e^{tx} f(x), \text{ if } x \text{ is discrete}$$

Thm If two 'RV's have the same MGF, they are of the same distribution.

Normal Distributions $X \sim N(\mu, \sigma^2)$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Eg) If the mgf of X is $e^{-7t + 10t^2}$, then $X \sim N(-7, 20)$

$$Z = \frac{X - \mu}{\sigma}, X \sim N(\mu, \sigma^2) \Rightarrow M_Z(t) = ?$$

Linear transformation : $Y = a \cdot X + b$, a&b are constants

$$M_Y(t) = E(e^{tY}) = E[e^{t(aX+b)}] = E(e^{atX+bt}) = E(e^{atX} \cdot e^{bt})$$