Math 711: Lecture of December 3, 2007

Step 6. Proof that J_+/J_* has finite length when d is minimum. We first prove that the test ideal commutes with localization for a reduced excellent Gorenstein local ring of prime characteristic $p > 0$. In order to do so, we introduce a new notion. Let (R, m, K) be a reduced excellent Gorenstein local ring of prime characteristic $p > 0$. We say that an ideal $J \subseteq R$ is *F-stable* provided that J is the annihilator of a submodule N of $H = H^d_m(R)$ that is stable under the action of F on H. We first note:

Lemma. *Let notation be as above.*

(a) *$J \subseteq R$ is F-stable if and only if for every ideal I of R generated by a system of parameters $x_1, \dots, x_d$, $(*)$ if $Ju \subseteq I$ then $Ju^p \subseteq I^{[p]}$. Moreover, for J to be F-stable, it suffices that for a single system of parameters $x_1, \dots, x_d$ for R, with $I_t = (x_1^t, \dots, x_d^t)R$ we have that $Ju \subseteq I_t$ implies that $Ju^p \subseteq I_t^{[p]}$ for all t.*

(b) *If J is F-stable and P is any prime ideal of R, then JR_P is F-stable.*

(c) *If J is an F-stable ideal of R that contains a nonzerodivisor, then $\tau(R) \subseteq J$.*

Proof. (a) Each element of H is represented by the class v of $u \in R$ in $R/I_t \hookrightarrow H$ for some t. The maps in the direct limit system for H are injective, and so J kills the class of u if and only if $Ju \subseteq I_t$. Then $F(v)$ is represented by v^p in R/I_{pt}, and $I_{pt} = I_t^{[p]}$. It is immediate both that condition $(*)$ for all I_t is necessary and sufficient for J to be F-stable, and since we may choose I to be any ideal generated by a system of parameters, we must have $(*)$ for all parameter ideals.

(b) Let $h = \text{height}\,(P)$ and choose a system of parameters $x_1, \dots, x_d \in m$ such that $x_1, \dots, x_h \in P$, and such that the images of $x_1, \dots, x_h$ form a system of parameters in R_P. It suffices to check that if $u/w \in R_P$, where $u \in R$ and $w \in R - P$, and $J(u/w)^p \in (x_1, \dots, x_h)R_P$, then $J(u/w) \in (x_1^p, \dots, x_h^p)R_P$. From the first condition we can choose $w' \in R - P$ such that $w'Ju \subseteq (x_1, \dots, x_h)R$, and the latter ideal is contained in $(x_1, \dots, x_h, x_{h+1}^N, \dots, x_d^N)R$ for all $N \geq 1$. Since J is F-stable, and $J(w'u) \subseteq (x_1, \dots, x_h, x_{h+1}^N, \dots, x_d^N)R$, we have that

$$J(w'u)^p \subseteq (x_1^p, \dots, x_h^p, x_{h+1}^{pN}, \dots, x_d^{pN})R$$

for all N. Intersecting the ideals on the right as N varies, we obtain that

$$(w')^p Ju^p \subseteq (x_1^p, \dots, x_h^p),$$

which implies that $J(u/w)^p \in (x_1^p, \dots, x_h^p)R_P$, as required.

(c) Let $c \in J \cap R^\circ$. Since c kills $N = \text{Ann}_H J$ if $v \in N$ we have that $v^q \in N$ for all e, and so $cv^q = 0$ for all q. It follows that $v \in 0^*_H$. Hence, $N \subseteq 0^*_H$, and so $\tau(R) = \text{Ann}_R 0^*_H \subseteq \text{Ann}(N) = \text{Ann}_R(\text{Ann}_H(J)) = J$. □

We shall need the following fact:

Proposition. *Let $(R,\, m,\, K)$ be an excellent reduced equidimensional local ring of prime characteristic $p > 0$, and let $d = \dim(R)$. Let $H = H^d_m(R)$. Then 0^*_H is stable under the action of F. If R is a domain, 0^+_H is stable under the action of F.*

Proof. Let $x_1,\, \ldots\,, x_d$ be a system of parameters. The first statement follows from the fact that if $u \in I^*_t$ for some t, then $u^p \in \bigl((I_t)^{[p]}\bigr)^*$. In fact, if $u \in I^*$ then $u^p \in (I^{[p]})^*$ in complete generality. The second assertion follows from the fact that if $u \in I^+_t$, then $u^p \in \bigl(I^{[p]}_t\bigr)^+$. The corresponding fact for any ideal I in any domain R follows from the fact that the Frobenius endomorphism on R^+ sends u to u^p and IR^+ to $I^{[p]}R^+$ while stabilizing R: hence, $u^p \in I^{[p]}R^+ \cap R$. $\square$

We next note:

Lemma. *Let R be a Noetherian ring of prime characteristic $p > 0$. Let $\mathfrak{A}$ be an ideal whose radical is contained only in maximal ideals of R, and let m be one maximal ideal of R. Then $(\mathfrak{A}R_m)^*$ in R_m is the same as $\mathfrak{A}^*R_m$.*

Proof. It suffices to prove $\subseteq$. Let $m = m_1,\, \ldots\,, m_k$ be the maximal ideals of R. If any m_i is also minimal, then $\{m_i\}$ is an isolated point of $\mathrm{Spec}\,(R)$, and the ring is a product. Every ideal is a product, and tight closure may be calculated separately in each factor. We can reduce to studying a factor where tthere are fewer maximal ideals. Therefore, we may assume that no m_i is minimal.

Then $\mathfrak{A}$ has primary decomposition $\mathfrak{A} = \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_k$ where $\mathfrak{A}_i$ is primary to m_i. Choose an element w of $\mathfrak{A}_2 \cap \cdots \cap \mathfrak{A}_k$ that is not in P, and not in any minimal prime of the ring.

Now suppose that $u/1 \in (IR_P)^*$ (we may clear denominators to assume the element has this form). By the Proposition on p. 2 of the Lecture Notes from September 17, we can choose $c \in R^\circ$ such that $cu^{[q]}/1 \in (\mathfrak{A}R_m)^{[q]}$ for all $q \gg 0$. Then $(*)\quad wc \in R^\circ \cap (R - m)$, and $c(wu)^q \in \mathfrak{A}^{[q]}$ for all $q \gg 0$. To see this, note that

$$\mathfrak{A}^{[q]} = (\mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_k)^{[q]} = (\mathfrak{A}_1 \cdots \mathfrak{A}_k)^{[q]}$$

(since the ideals $\mathfrak{A}_1,\, \ldots\,, \mathfrak{A}_k$ are pairwise comaximal). This becomes $\mathfrak{A}^{[q]}_1 \cap \cdots \cap \mathfrak{A}^{[q]}_k$ and, since the ideal m_i is maximal, the ideal $\mathfrak{A}^{[q]}_i$ is primary to m_i. Then cu^q is in the contraction of $(\mathfrak{A}R_m)^{[q]}$ to R, and this is $\mathfrak{A}^{[q]}_1$, while $w^q \in \mathfrak{A}^{[q]}_i$ for $i > 1$. This proves $(*)$, and, hence, $wu \in I^*$ and $u \in W^{-1}I^*$. $\square$

Theorem (K. E. Smith). *Let $(R,\, m,\, K)$ be an excellent reduced Gorenstein local ring of prime characteristic $p > 0$. Let P be a prime ideal of R. Then $\tau(R_P) = \tau(R)_P$.*

Proof. We know that both ideals are generated by nonzerodivisors. We first show that $\tau(R)_P \subseteq \tau(R_P)$. Let $c \in \tau(R)$. Let $\mathrm{height}\,(P) = h$ and let $x_1,\, \ldots\,, x_h$ be part of a system of parameters for R whose images in R_P give a system of parameters for R_P. Let $\mathfrak{A}_t = (x^t_1,\, \ldots\,, x^t_h)R$. By part (d) of the Theorem on p. 5 of the Lecture Notes from November 30, it suffices to show that for every t, $c(\mathfrak{A}_t R_P)^*_{R_P} \subseteq \mathfrak{A}_t R_P$. We claim that $(\mathfrak{A}_t R_P)^* = \mathfrak{A}_t^* R_P$. By Problem 2(a) of Problem Set #3, we can localize at the multiplicative system W which is the complement of the union of the minimal primes of $\mathfrak{A}$, since elements of W are nonzerodivisors on every $\mathfrak{A}^{[q]}$. In the resulting semilocal ring, the expansion of P is maximal, and we may apply the preceding Lemma to obtain that $(\mathfrak{A}_t R_P)^* = \mathfrak{A}_t^* R_P$. But then $c\mathfrak{A}_t^* \subseteq \mathfrak{A}_t$, and so $c(\mathfrak{A}_t R_P)^* = c\mathfrak{A}_t^* R_P \subseteq \mathfrak{A}_t R_P$, as required.

To prove the other direction, let d be the Krull dimension of R and let $H = H^d_m R)$. Then the annihilator of 0^*_H in R is $\tau(R)$. Hence, by the Proposition above, $\tau(R)$ is an F-stable ideal. It follows that $\tau(R)R_P$ is an F-stable ideal of R_P by part (b) of the Lemma on p. 1. It contains a nonzerodivisor, since $\tau(R)$ does. By part (c) of the Lemma on p. 1, $\tau(R_P) \subseteq \tau(R)R_P$. $\square$

We can now prove:

Lemma. *Let* (R, m, K) *be a complete local Gorenstein domain of Krull dimension* d *of prime characteristic* $p > 0$ *such that, for* $h < d$, *tight closure is the same as plus closure for ideals generated by* h *elements that are part of a system of parameters. Then* J_+/J_* *has finite length. Hence,* $0^*/0^+$ *has finite length.*

Proof. Since J_+/J_* is finitely generated, it suffices to prove that it becomes 0 when we localize at a prime ideal P of R strictly contained in m. Since $J_* = \tau(R)$, we have that $(J_*)_P = J_* R_P = \tau(R_P)$, by the Theorem above. Hence, it suffices to prove that every element of J_+ maps to a test element in R_P. Let $c \in J_+$. Let $h = \operatorname{height}(P)$. Let $x_1, \ldots, x_d$ be a system of parameters for R such that $x_1, \ldots, x_h \in P$ and their images in R_P are a system of parameters for R_P. Then it suffices to show that

$$c\big((x_1^t, \ldots, x_h^t)R_P\big)^* \subseteq (x_1^t, \ldots, x_h^t)R_P$$

for all t. We have that

$$\big((x_1^t, \ldots, x_h^t)R_P\big)^* = (x_1^t, \ldots, x_h^t)^* R_P$$

and

$$\big((x_1^t, \ldots, x_h^t)R\big)^* = \big((x_1^t, \ldots, x_h^t)R\big)^+ \subseteq \big((x_1^t, \ldots, x_h^t, x_{h+1}^N, \ldots, x_d^N)R\big)^+$$

for all $N \geq 1$. Since $c \in J_+$, this yields

$$c\big((x_1^t, \ldots, x_h^t)R_P\big)^* \subseteq (x_1^t, \ldots, x_h^t, x_{h+1}^N, \ldots, x_d^N)R$$

for all $N \geq 1$. We may intersect the ideals on the right as N varies to obtain

$$c\big((x_1^t, \ldots, x_h^t)R_P\big)^* \subseteq (x_1^t, \ldots, x_h^t)R,$$

and localizing at P then gives the result that we require. $\square$