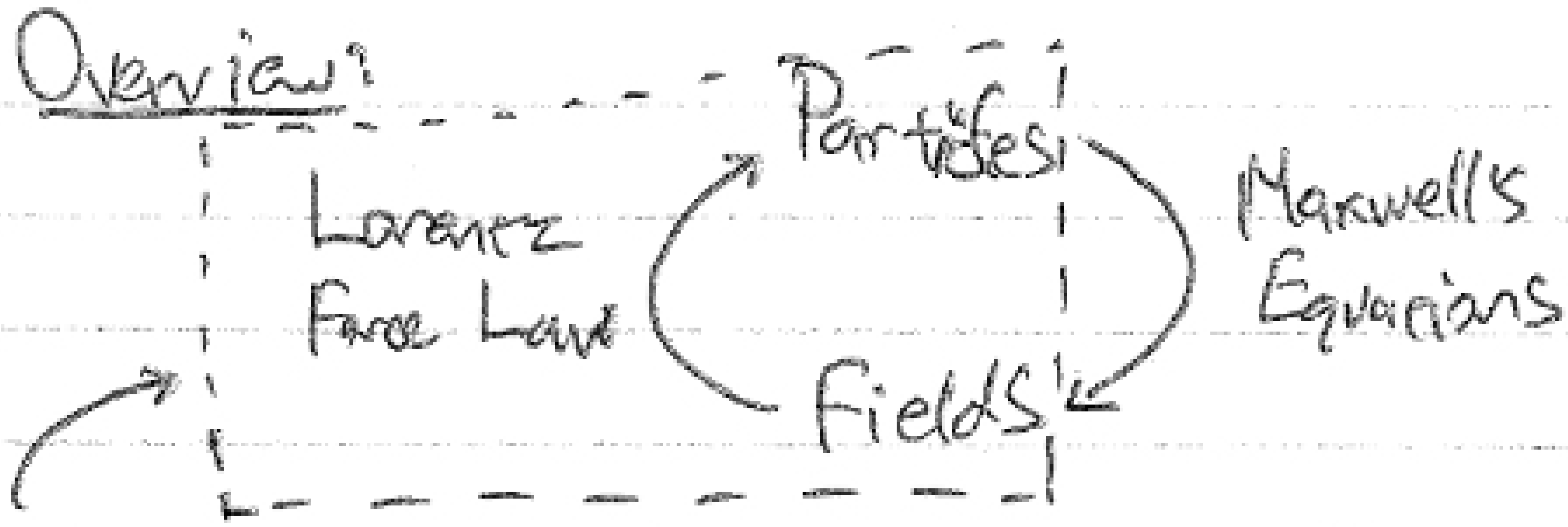
Lecture #3: Single Particle Motion in Constant Uniform Fields — Howes ①

I. Overview:



Today, and for next few weeks, we will focus on "half" of the plasma physics problem, the inconsistent picture of the effect of fields on the motion of particles, neglected the particles effect on fields.

II. Constant, Uniform $\underset{\sim}{B}$ with $\underset{\sim}{E}=0$.

A. 1. Nonrelativistic Limit

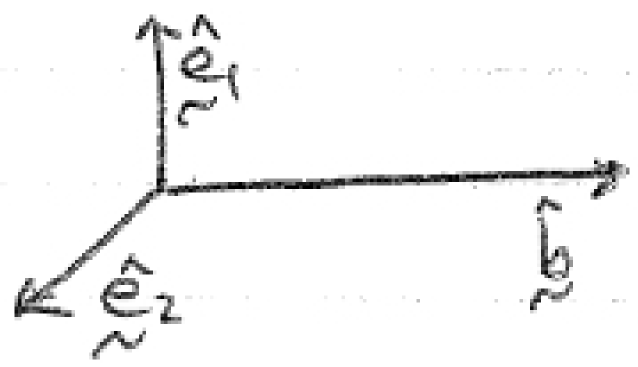
2. Lorentz Force Law $\quad m_s \frac{d\underset{\sim}{v}_s}{dt} = q_s\left(\underset{\sim}{E} + \underset{\sim}{v}_s \times \underset{\sim}{B}\right)$

3. Drop species subscript "s" because single particle motion does not depend on other species (or any other particles).

4. Thus, for $\underset{\sim}{E}=0$, $\quad m\frac{d\underset{\sim}{v}}{dt} = q\,\underset{\sim}{v}\times\underset{\sim}{B}$

B. Solve

1. Define $\hat{\underset{\sim}{b}} \equiv \frac{\underset{\sim}{B}}{|\underset{\sim}{B}|}$ and unit vectors $\hat{\underset{\sim}{e}}_1$ and $\hat{\underset{\sim}{e}}_2$ to define a right-handed, orthonormal basis



$\hat{\underset{\sim}{e}}_1 \times \hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{b}}$

2. a. We can write $\underset{\sim}{v} = v_1\hat{\underset{\sim}{e}}_1 + v_2\hat{\underset{\sim}{e}}_2 + v_\parallel\hat{\underset{\sim}{b}}$,

and $\underset{\sim}{B} = B_0\hat{\underset{\sim}{b}}$

b. Thus $\quad \frac{d\underset{\sim}{v}}{dt} = \frac{qB_0}{m}\,\underset{\sim}{v}\times\hat{\underset{\sim}{b}}$

c. NOTE: $\underset{\sim}{v}\times\hat{\underset{\sim}{b}} = -v_1\hat{\underset{\sim}{e}}_2 + v_2\hat{\underset{\sim}{e}}_1$

II.B. (Continued)

3. Take dot product with each unit vector to get component equations

$\hat{\underset{\sim}{e}}_1 \cdot \quad \frac{d\underset{\sim}{v}}{dt} = \frac{qB_0}{m} \underset{\sim}{v} \times \hat{\underset{\sim}{b}} \quad \Rightarrow \quad \frac{dv_1}{dt} = \frac{qB_0}{m} v_2$

$\hat{e}_2 \cdot \quad$ " $\quad \Rightarrow \quad \frac{dv_2}{dt} = -\frac{qB_0}{m} v_1$

$\hat{\underset{\sim}{b}} \cdot \quad$ " $\quad \Rightarrow \quad \frac{dv_\parallel}{dt} = 0$

4. Take $\frac{d}{dt}$ of $\hat{\underset{\sim}{e}}_1$ eq. $\quad \frac{d^2 v_1}{dt^2} = \frac{qB_0}{m}\left(\frac{dv_2}{dt}\right) \overset{\text{substitute } \hat{\underset{\sim}{e}}_2 \text{ eq.}}{=} -\left(\frac{qB_0}{m}\right)^2 v_1$

This is just the equation of a <u>simple harmonic oscillator</u>.

We define the $\boxed{\text{Cyclotron Frequency} \quad \omega_c \equiv \frac{qB_0}{m}}$

Thus, $\boxed{\frac{d^2 v_1}{dt^2} = -\omega_c^2 v_1}$

5. Other properties: a. Take $\underset{\sim}{v} \cdot$ with equation:

$$\underset{\sim}{v} \cdot \frac{d\underset{\sim}{v}}{dt} = \frac{1}{2}\frac{dv^2}{dt} = 0$$

b. So $v^2 = \sqrt{v_1^2 + v_2^2 + v_\parallel^2}$ is constant

c. But, we also know from $\hat{\underset{\sim}{b}}$ equation that $\frac{dv_\parallel}{dt} = 0$, so

$\boxed{v_\perp \equiv \sqrt{v_1^2 + v_2^2}}$ is constant.

d. Acceleration is perpendicular to $\underset{\sim}{v}$, so Lorentz Force cannot change the energy of the particles.

6. <u>Complete Solution</u>: a. General Solution $v_1 = Ae^{-i\omega_c t} + Be^{i\omega_c t}$

b. $\hat{\underset{\sim}{e}}_1$ component gives $v_2 = \pm i(-Ae^{-i\omega_c t} + Be^{i\omega_c t})$

c. Take initial conditions $\begin{matrix} v_{10} = v_\perp \\ v_{20} = 0 \end{matrix}$ at time $t=0$.

II. B. 6. (Continued)

d. This gives $A = B = \frac{v_\perp}{2}$

e. Solution for velocity: $v_1 = \frac{v_\perp}{2}\left(e^{i\omega_c t} + e^{-i\omega_c t}\right) = v_\perp \cos\omega_c t$

$$v_2 = \frac{i v_\perp}{2}\left(e^{i\omega_c t} - e^{-i\omega_c t}\right) = v_\perp \sin\omega_c t$$

$$v_\parallel = v_{\parallel 0}$$

7. Solve for Position $\underline{x}$.

a. $v_1 = \frac{dx_1}{dt}$ so $x_1 = \frac{v_\perp}{\omega_c}\sin\omega_c t + x_{10}$

$v_2 = \frac{dx_2}{dt}$ $\quad x_2 = \frac{v_\perp}{\omega_c}\cos\omega_c t + x_{20}$

$v_\parallel = \frac{dx_\parallel}{dt}$ $\quad x_\parallel = v_{\parallel 0} t + x_{\parallel 0}$

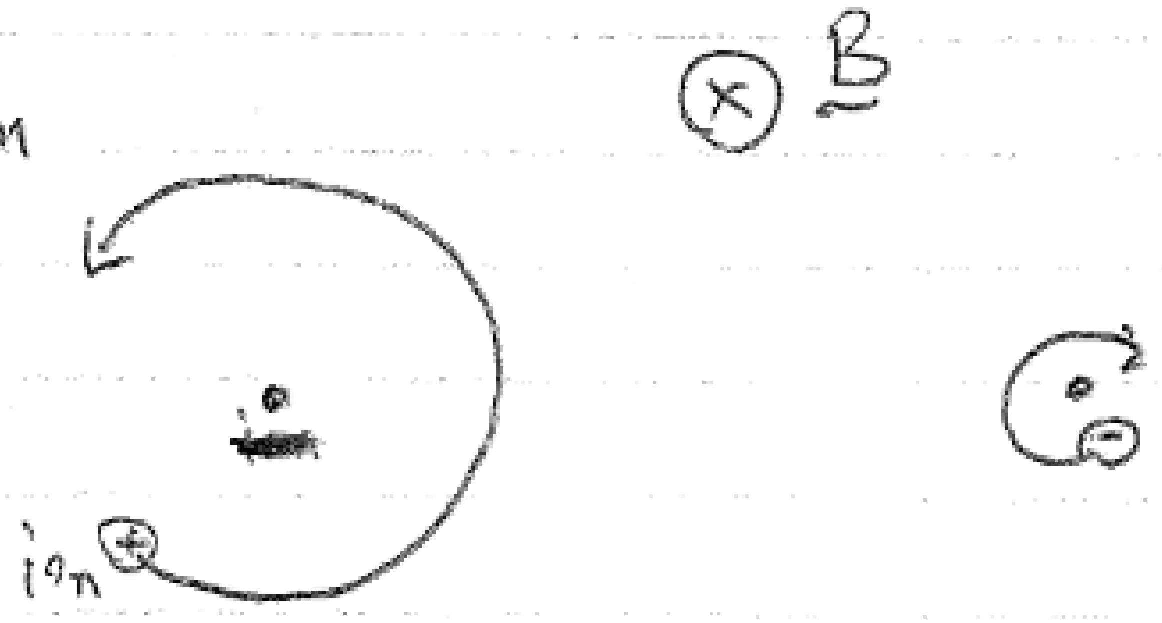
b. We can define the Larmor radius (note this is not the thermal Larmor radius defined using v_t)

$$\boxed{r_L \equiv \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{q B_0}}$$

c. The perpendicular components x_1 & x_2, can then be written.

$$x_1 = r_L \sin\omega_c t + x_{10}$$
$$x_2 = r_L \cos\omega_c t + x_{20}$$

8. Larmor Motion



Note the the Magnetic field created by Larmor Motion opposes the mean field

⇒ DIAMAGNETIC behavior