Math 711: Lecture of September 5, 2007

Throughout these lecture notes all given rings are assumed commutative, associative, with identity and modules are assumed unital. Homomorphisms are assumed to preserve the identity. With a few exceptions that will be noted as they occur, given rings are assumed to be Noetherian. However, we usually include this hypothesis, especially in formal statements of theorems.

Our objective is to discuss tight closure closure theory and its connection with the existence of big Cohen-Macaulay algebras, as well as the applications that each of these have: they have many in common.

At certain points in these notes we will include material not covered in class that we want to assume. We indicate where such digressions begin and end with double bars before and after, just as we have done for these two paragraphs. On first perusal, the reader may wish to read only the unfamiliar definitions and the statements of theorems given, and come back to the proofs later.

In particular, the write-up of this first lecture is much longer than will be usual, since a substantial amount of prerequisite material is explained, often in detail, in this manner.

By a *quasilocal ring* $(R,\, m,\, K)$ we mean a ring with a unique maximal ideal m: in this notation, $K = R/m$. A quasilocal ring is called *local* if it is Noetherian. A homomorphism $h : R \to S$ from a quasilocal ring $(R,\, m,\, K)$ to a quasilocal ring $(S,\, m_S,\, K_S)$ is called *local* if $h(m) \subseteq m_S$, and then h induces a map of residue fields $K \to K_S$.

If $x_1, \ldots, x_n \in R$ and M is an R-module, the sequence $x_1, \ldots, x_n$ is called a *possibly improper* regular sequence on M if x_1 is not a zerodivisor on M and for all i, $0 \le i \le n-1$, x_{i+1} is not a zerodivisor on $M/(x_1, \ldots, x_i)M$. A possibly improper regular sequence is called a *regular sequence* on M if, in addition, $(*)\quad (x_1, \ldots, x_n)M \neq M$. When $(*)$ fails, the regular sequence is called *improper*. When $(*)$ holds we may say that the regular sequence is *proper* for emphasis, but this use of the word “proper” is not necessary.

Note that every sequence of elements is an improper regular sequence on the 0 module, and that a sequence of any length consisting of the element 1 (or units of the ring) is an improper regular sequence on every module.

If $x_1, \ldots, x_n \in m$, the maximal ideal of a local ring $(R,\, m,\, K)$, and M is a nonzero finitely generated R-module, then it is automatic that if $x_1, \ldots, x_n$ is a possibly improper regular sequence on M then $x_1, \ldots, x_n$ is a regular sequence on M: we know that $mM \neq M$ by Nakayama's Lemma.

If $x_1, \dots, x_n \in R$ is a possibly improper regular sequence on M and and S is any flat R-algebra, then the images of $x_1, \dots, x_n$ in S form a possibly improper regular sequence on $S \otimes_R M$. By a straightforward induction on n, this reduces to the case where $n = 1$, where it follows from the observation that if $0 \to M \to M$ is exact, where the map is given by multiplication by x, this remains true when we apply $S \otimes_R \_$. In particular, this holds when S is a localization of R.

If $x_1, \dots, x_n$ is a regular sequence on M and S is flat over R, it remains a regular sequence provided that $S \otimes_R \big(M/(x_1, \dots, x_n)M\big) \neq 0$, which is always the case when S is faithfully flat over R.

Nakayama's Lemma, including the homogeneous case

Recall that in Nakayama's Lemma one has a *finitely generated module* M over a quasilocal ring (R, m, K). The lemma states that if $M = mM$ then $M = 0$. (In fact, if $u_1, \dots, u_h$ is a set of generators of M with h minimum, the fact that $M = mM$ implies that $M = mu_1 + \cdots mu_h$. In particular, $u_h = f_1u_1 + \cdots + f_hu_h$, and so $(1 - f_h)u_h = f_1u_1 + \cdots + f_{h-1}u_{h-1}$ (or 0 if $h = 1$). Since $1 - f_h$ is a unit, u_h is not needed as a generator, a contradiction unless $h = 0$.)

By applying this result to M/N, one can conclude that if M is finitely generated (or finitely generated over N), and $M = N + mM$, then $M = N$. In particular, elements of M whose images generate M/mM generate M: if N is the module they generate, we have $M = N + mM$. Less familiar is the homogeneous form of the Lemma: it does not need M to be finitely generated, although there can be only finitely many negative graded components (the detailed statement is given below).

First recall that if H is an additive semigroup with 0 and R is an H-graded ring, we also have the notion of an H-graded R-module M: M has a direct sum decomposition

$$M = \bigoplus_{h \in H} M_h$$

as an abelian group such that for all $h, k \in H$, $R_hM_k \subseteq M_{h+k}$. Thus, every M_h is an R_0-module. A submodule N of M is called graded (or homogeneous) if

$$N = \bigoplus_{h \in H} (N \cap M_h).$$

An equivalent statement is that the homogeneous components in M of every element of N are in N, and another is that N is generated by forms of M.

Note that if we have a subsemigroup $H \subseteq H'$, then any H-graded ring or module can be viewed as an H'-graded ring or module by letting the components corresponding to elements of $H' - H$ be zero.

In particular, an $\mathbb{N}$-graded ring is also $\mathbb{Z}$-graded, and it makes sense to consider a $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded ring.

Nakayama's Lemma, homogeneous form. *Let R be an $\mathbb{N}$-graded ring and let M be any $\mathbb{Z}$-graded module such that $M_{-n} = 0$ for all sufficiently large n (i.e., M has only finitely many nonzero negative components). Let I be the ideal of R generated by elements of positive degree. If $M = IM$, then $M = 0$. Hence, if N is a graded submodule such that $M = N + IM$, then $N = M$, and a homogeneous set of generators for M/IM generates M.*

Proof. If $M = IM$ and $u \in M$ is nonzero homogeneous of smallest degree d, then u is a sum of products $i_t v_t$ where each $i_t \in I$ has positive degree, and every v_t is homogeneous, necessarily of degree $\geq d$. Since every term $i_t v_t$ has degree strictly larger than d, this is a contradiction. The final two statements follow exactly as in the case of the usual form of Nakayama's Lemma. $\square$

In general, regular sequences are not permutable: in the polynomial ring $R = K[x, y, z]$ over the field K, $x-1, xy, xz$ is a regular sequence but $xy, xz, x-1$ is not. However, if M is a finitely generated nonzero module over a local ring (R, m, K), a regular sequence on M is permutable. This is also true if R is $\mathbb{N}$-graded, M is $\mathbb{Z}$-graded but nonzero in only finitely many negative degrees, and the elements of the regular sequence in R have positive degree.

To see why, note that we get all permutations if we can transpose two consecutive terms of a regular sequence. If we kill the ideal generated by the preceding terms times the module, we come down to the case where we are transposing the first two terms. Since the ideal generated by these two terms does not depend on their order, it suffices to consider the case of regular sequences x, y of length 2. The key point is to prove that y is not a zerodivisor on M. Let $N \subseteq M$ by the annihilator of y. If $u \in N$, $yu = 0 \in xM$ implies that $u \in xM$, so that $u = xv$. Then $y(xv) = 0$, and x is not a zerodivsior on M, so that $yv = 0$, and $v \in N$. This shows that $N = xN$, contradicting Nakayama's Lemma (the local version or the homogeneous version, whichever is appropriate).

The next part of the argument does not need the local or graded hypothesis: it works quite generally. We need to show that x is a nonzerodivisor on M/yM. Suppose that $xu = yv$. Since y is a nonzerodivisor on xM, we have that $v = xw$, and $xu = yxw$. Thus $x(u - yw) = 0$. Since x is a nonzerodivisor on M, we have that $u = yw$, as required. $\square$

The Krull dimension of a ring R may be characterized as the supremum of lengths of chains of prime ideals of R, where the length of the strictly ascending chain

$$P_0 \subset P_1 \subset \cdots \subset P_n$$

is n. The Krull dimension of the local ring (R, m, K) may also be characterized as the least integer n such that there exists a sequence $x_1, \ldots, x_n \in m$ such that $m =$