Generalized Permutations and Combinations

- Sometimes you have identical things in a set. Of course it really doesn't count as a set then, because officially repeated elements don't occur. As a mathematical object, the new kind of set is sometimes called a *multiset*.
- Example: a set of 4 red balls and 3 green balls: $\{r, r, r, r, g, g, g\}$.
- In case things are ordered, we've already dealt with this, as in (r, r, r, r, g, g, g). This is just a sequence of g's and r's.
- All of this section deals with allowing repetition, but differing kinds.
- Example: how many ways can you rearrange the word MISSISSIPPI?

Sampling with replacement

- There is a drawing for 3 prizes: a Yugo, an IBM PC, and a Kelvinator. There are seven contestants: George, Fred, Luke, Eileen, Kerith, Metta, and Al. In each drawing, any of the seven contestants can win the prize, and, moreover, they are still eligible in the next drawing. How many possible outcomes are there? An outcome is "The Yugo and the Kelvinator went to George and the PC to Kerith."
- The book has a thing called an r-permutation of n objects. This basically consists of filling r named slots (the prizes) with any of n objects (the contestants), where the objects going into the slots can be reused. Here there is no restriction on how big r can be.
- One can think of this as sampling from an n-element population r times, with replacement allowed, and where we care the outcome of r samplings is a record of the elements in a list of length r.
- I don't like the use of the word "r-permutation" here, as there isn't any permuting or rearrangement. I like "r-sample with replacement." A permutation of r out of n objects is then an "r-sample without replacement".
- Anyhow, the number of r-permutations of n objects is just n^r, by the product rule. There are 7^3 possible outcomes.

Sampling with replacement, order doesn't matter

- In this situation, we look at the same contestants, but an outcome is just "George won 2 prizes and Kerith won 1". Here we're selecting George twice and Kerith once. We are forming a 3-element multiset out of a 7-element set. Or, turning things around, the prizes are indistinguishable, and we're distributing them to the seven contestants.

- How do we count the number of outcomes in this case?

- The outcome above can be written $2g + 1k$, 2 georges and one kerith. It can also be written $2g + 1k + 0f + 0l + 0m + 0e + 0a$. Even more stylistically it can be written xx+x++++, if we regard the seven people as given in the fixed order george, kerith, fred, luke, metta, eileen, al. In this form there is a string of length 9, three of which are x's. Any such string represents a possible outcome; the string ++++x+xx+ represents 0 george, 0 kerith. 0 fred. 0 luke, 1metta, 2 eileen, and 0 al. The number of outcomes is then $\binom{9}{3}$.

- You can also get this result by going back to the "distribution of identical prizes" way of proceeding. Imagine that each contestant has a box, which is initially empty. Represent the 7 empty boxes by the string ++++++ of 6 plus signs. (George's box is at the left of all the plus signs and Al's is at the right of them.) Then three x's (the prizes) are thrown at the boxes. Each prize nestles into the space between the characters ,and expands the string. So after we throw out the first prize x, the string might look like ++++x++, where the prize fell into the 5th box. Then we might get x++++x++, where the next prize went into the first box.

- In general, distributing r identical prizes to n people can be done in $\binom{n-1+r}{r}$ ways.