Math 711: Lecture of September 24, 2007

Flat base change and Hom

We want to discuss in some detail when a short exact sequence splits. The following result is very useful.

Theorem (Hom commutes with flat base change). *If S is a flat R-algebra and M, N are R-modules such that M is finitely presented over R, then the canonical homomorphism*

$$\theta_M : S \otimes_R \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

sending $s \otimes f$ to $s(\mathbf{1}_S \otimes f)$ is an isomorphism.

Proof. It is easy to see that θ_R is an isomorphism and that $\theta_{M_1 \oplus M_2}$ may be identified with $\theta_{M_1} \oplus \theta_{M_2}$, so that θ_G is an isomorphism whenever G is a finitely generated free R-module.

Since M is finitely presented, we have an exact sequence $H \to G \twoheadrightarrow M \to 0$ where G, H are finitely generated free R-modules. In the diagram below the right column is obtained by first applying $S \otimes_R \_$ (exactness is preserved since $\otimes$ is right exact), and then applying $\mathrm{Hom}_S(\_, S \otimes_R N)$, so that the right column is exact. The left column is obtained by first applying $\mathrm{Hom}_R(\_, N)$, and then $S \otimes_R \_$ (exactness is preserved because of the hypothesis that S is R-flat). The squares are easily seen to commute.

$$\begin{array}{ccc}
S \otimes_R \mathrm{Hom}_R(H, N) & \xrightarrow{\theta_H} & \mathrm{Hom}_S(S \otimes_R H, S \otimes_R N) \\
\uparrow & & \uparrow \\
S \otimes_R \mathrm{Hom}_R(G, N) & \xrightarrow{\theta_G} & \mathrm{Hom}_S(S \otimes_R G, S \otimes_R N) \\
\uparrow & & \uparrow \\
S \otimes_R \mathrm{Hom}_R(M, N) & \xrightarrow{\theta_M} & \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \\
\uparrow & & \uparrow \\
0 & \longrightarrow & 0
\end{array}$$

From the fact, established in the first paragraph, that θ_G and θ_H are isomorphisms and the exactness of the two columns, it follows that θ_M is an isomorphism as well (kernels of isomorphic maps are isomorphic). $\square$

Corollary. *If W is a multiplicative system in R and M is finitely presented, we have that* $W^{-1}\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$.

Moreover, if (R, m) is a local ring and both M, N are finitely generated, we may identify $\mathrm{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$ *with the m-adic completion of* $\mathrm{Hom}_R(M, N)$ *(since m-adic completion is the same as tensoring over R with $\widehat{R}$ (as covariant functors) on finitely generated R-modules).* □

When does a short exact sequence split?

Throughout this section, $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ is a short exact sequence of modules over a ring R. There is no restriction on the characteristic of R. We want to discuss the problem of when this sequence splits. One condition is that there exist a map $\eta : M \to N$ such that $\eta\alpha = \mathbf{1}_N$. Let $Q' = \mathrm{Ker}\,(\eta)$. Then Q' is disjoint from the image $\alpha(N) = N'$ of N in M, and $N' + Q' = M$. It follows that M is the internal direct sum of N' and Q' and that β maps Q' isomorphically onto Q.

Similarly, the sequence splits if there is a map $\theta : Q \to M$ such that $\beta\theta = \mathbf{1}_Q$. In this case let $N' = \alpha(N)$ and $Q' = \theta(Q)$. Again, N' and Q' are disjoint, and $N' + Q' = M$, so that M is again the internal direct sum of N' and Q'.

Proposition. *Let R be an arbitrary ring and let*

$$(\#) \quad 0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$$

be a short exact sequence of R-modules. Consider the sequence

$$(*) \quad 0 \to \mathrm{Hom}_R(Q,\, N) \xrightarrow{\alpha_*} \mathrm{Hom}_R(Q,\, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Q,\, Q) \to 0$$

which is exact except possibly at $\mathrm{Hom}_R(Q, Q)$*, and let* $C = \mathrm{Coker}\,(\beta_*)$*. The following conditions are equivalent:*

(1) *The sequence $(\#)$ is split.*

(2) *The sequence $(*)$ is exact.*

(3) *The map β_* is surjective.*

(4) $C = 0$.

(5) *The element $\mathbf{1}_Q$ is in the image of β_*.*

Proof. Because Hom commutes with finite direct sum, we have that (1) ⇒ (2), while (2) ⇒ (3) ⇔ (4) ⇒ (5) is clear. It remains to show that (5) ⇒ (1). Suppose $\theta : Q \to M$ is such that $\beta_*(\theta) = \mathbf{1}_Q$. Since β_* is induced by composition with β, we have that $\beta\theta = \mathbf{1}_Q$. □

A split exact sequence remains split after any base change. In particular, it remains split after localization. There are partial converses. Recall that if $I \subseteq R$,

$$\mathcal{V}(I) = \{P \in \mathrm{Spec}\,(R) : I \subseteq P\},$$

and that

$$\mathcal{D}(I) = \mathrm{Spec}\,(R) - \mathcal{V}(I).$$

In particular,

$$\mathcal{D}(fR) = \{P \in \mathrm{Spec}\,(R) : f \notin P\},$$

and we also write $\mathcal{D}(f)$ or $\mathcal{D}_f$ for $\mathcal{D}(fR)$.

Theorem. *Let R be an arbitrary ring and let*

$$(\#) \quad 0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$$

be a short exact sequence of R-modules such that Q is finitely presented.

(a) *(#) is split if and only if for every maximal ideal m of R, the sequence*

$$0 \to N_m \to M_m \to Q_m \to 0$$

is split.

(b) *Let S be a faithfully flat R-algebra. The sequence (#) is split if and only if the sequence*

$$0 \to S \otimes_R N \to S \otimes_R M \to S \otimes_R Q \to 0$$

is split.

(c) *Let W be a multiplicative system in R. If the sequence*

$$0 \to W^{-1}N \to W^{-1}M \to W^{-1}Q \to 0$$

is split over $W^{-1}R$, then there exists a single element $c \in W$ such that

$$0 \to N_c \to M_c \to Q_c \to 0$$

is split over R_c.

(d) *If P is a prime ideal of R such that*

$$0 \to N_P \to M_P \to Q_P \to 0$$

is split, there exists an element $c \in R - P$ such that

$$0 \to N_c \to M_c \to Q_c \to 0$$

is split over R_c. Hence, (#) becomes split after localization at any prime P' that does not contain c, i.e., any prime P' such that $c \notin P'$.

(e) *The split locus for (#), by which we mean the set of primes $P \in \mathrm{Spec}\,(R)$ such that*

$$0 \to N_P \to M_P \to Q_P \to 0$$