Math 711: Lecture of November 5, 2007

The following result is one we have already established in the F-finite case. We can now extend it to include rings essentially of finite type over an excellent semilocal ring.

Theorem. *Let R be a reduced ring of prime characteristic $p > 0$ essentially of finite type over an excellent semilocal ring B. Suppose that $c \in R^\circ$ is such that R_c is strongly F-regular. Then c has a power that is a completely stable big test element in R.*

Proof. If c is a completely stable big test element in a faithfully flat extension of R, then that is also true for R by part (b) of the Proposition at the bottom of p. 8 of the Lecture Notes from September 17.

The hypothesis continues to hold if we replace R by $\widehat{B} \otimes_B R$, and it holds in each factor of this ring. We may therefore assume that R is essentially of finite type over a complete local ring A. As usual, choose a coefficient field K for A and a p-base Λ for K. Again, the hypotheis continues to hold is we replace R by R^Γ for $\Gamma \ll \Lambda$, and R^Γ is faithfully flat over R. But now we are done, since R^Γ is F-finite. $\square$

We next want to backtrack and prove that certain rings are approximately Gorenstein in a much simpler way than in the lengthy and convoluted argument given in the Lecture Notes from October 24. While the result we prove is much weaker, it does suffice for the case of an excellent normal Cohen-Macaulay ring, and, hence, for excellent weakly F-regular rings.

We first note:

Lemma. *Let M and N be modules over a Noetherian ring R and let x be a nonzerodivisor on N. Suppose that M is R-free or, much more generally, that $\mathrm{Ext}^1_R(M, N) = 0$. Then*

$$(R/xR) \otimes_R \mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{R/xR}(M/xM, N/xN).$$

Proof. The right hand module is evidently the same as $\mathrm{Hom}_R(M/xM, N/xN)$, and also the same as $\mathrm{Hom}_R(M, N/xN)$, since any map $M \to N/xN$ must kill xM. Apply $\mathrm{Hom}_R(M, \_)$ to the short exact sequence

$$0 \to N \xrightarrow{x\cdot} N \to N/xN \to 0.$$

This yields a long exact sequence which is, in part,

$$0 \to \mathrm{Hom}_R(M, N) \xrightarrow{x\cdot} \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N/xN) \to \mathrm{Ext}^1_R(M, N) = 0,$$

and the result follows. $\square$

Theorem. *Let (R, m, K) be an excellent, normal, Cohen-Macaulay ring, or, more generally, any Cohen-Macaulay local ring whose completion is a Cohen-Macaulay local domain. Then R is approximately Gorenstein.*

Proof. We may replace R by its completion and then R is module-finite over a regular local ring $A \subseteq R$. Because R is Cohen-Macaulay, it is free of some rank h as an A-module, i.e., $R \cong A^h$. Then $\omega = \mathrm{Hom}_A(R, A)$ is also an R-module, and is also isomorphic to A^h as an A-module. (We shall see later that ω is what is called a *canonical module* for R. Up to isomorphism, it is independent of the choice of A.) Then ω is, evidently, also a Cohen-Macaulay module over R. We want to see that it has type one. This only uses the Cohen-Macaulay property of R: it does not use the fact that R is a domain.

From the Lemma above, we see that the calculation of ω commutes with killing a parameter in A. We may choose a system of parameters for A (and R) that is a minimal set of generators for the maximal ideal of A. By killing these one at a time, we reduce to seeing this when $A = K$ is a field and R is a zero-dimensional local ring with coefficient field K. We claim that in this case, $\omega = \mathrm{Hom}_K(R, K)$ is isomorphic with $E_R(K)$. In fact, ω is injective because for any R-module M,

$$\mathrm{Hom}_R(M, \omega) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_K(R, K)) \cong \mathrm{Hom}_K(M \otimes_R R, K) \cong \mathrm{Hom}_K(M, K)$$

by the adjointness of tensor and Hom. This is, in fact, a natural isomorphism of functors. Since $\mathrm{Hom}_K(\_, K)$ is exact, so is $\mathrm{Hom}_R(\_, \omega)$. Thus, ω is a direct sum of copies of $E_R(K)$. But its length is the same as its dimension as a K-vector space, and this is the same as the dimension of R as a K-vector space, which is the length of R. Thus, ω has the same length as $E_R(K)$, and it follows that $\omega \cong E_R(K)$.

We now return to the situation where R is a domain. Since every nonzero element of R has a nonzero multiple in A, we have that ω is torsion-free as an R-module. Thus, if w is any nonzero element of ω, we have an embedding $R \to \omega$ sending $1 \mapsto w$. Let $I_t = (x_1^t, \ldots, x_n^t)$, where $x_1, \ldots, x_n$ is a system of parameters for R. Then $I_t\omega \cap Rw$ must have the form $J_t w$ for some m-primary ideal J_t of R. Then

$$R/J_t \cong Rw/(I_t\omega \cap Rw) \subseteq \omega/I_t\omega.$$

Since $\omega/I_t\omega$ is an injective hull of the residue class field for R/I_t, it is an essential extension of its socle. Therefore, R/J_t is an essential extension of its socle as well. Consequently, $J_t \subseteq R$ is irreducible and m-primary. It will now suffice to show that the ideals J_t are cofinal with the powers of m.

By the Artin-Rees Lemma there exists a constant integer $a \in \mathbb{N}$ such that

$$m^{N+a}\omega \cap Rw \subseteq m^N(Rw) = m^N w$$

for all N. But then $J_{N+a} \subseteq m^n$, since $I_{N+a} \subseteq m^{N+a}$. $\square$

We next want to prove some additional results on openness of loci, such as the Cohen-Macaulay locus. The following fact is very useful.

Lemma on openness of loci. *Let $X = \operatorname{Spec}(R)$, where R is a Noetherian ring. Then $S \subseteq X$ is open if and only if the following two conditions hold:*

(1) *If $P \subseteq Q$ and $Q \in S$ then $P \in S$.*

(2) *For all $P \in S$, $S \cap \mathcal{V}(P)$ is open in $\mathcal{V}(P)$.*

The second condition can be weakened to:

(2°) *For all $P \in S$, S contains an open neighborhood of P in $\mathcal{V}(P)$.*

Proof. It is clear that (1), (2), and (2°) are necessary for S to be open. Since (2°) is weaker than (2), and it suffices to show that (1) and (2°) imply that S is open. Suppose otherwise. Since R has DCC on prime ideals, if S is not open there exists a minimal element P of S that has no open neighborhood entirely contained in S. For all primes Q strictly contained in P, choose an open neighborhood U_Q of Q contained entirely in S. Let U be the union of these open sets: the U is an open set contained entirely in S, and contains all primes Q strictly smaller than P.

Let $Z = X - U$, which is closed. It follows that Z has finitely many minimal elements, one of which must be P. Call them $P = P_0, P_1, \dots, P_k$. Then

$$Z = \mathcal{V}(P_0) \cup \cdots \mathcal{V}(P_k).$$

Finally, choose U' open in X such that $P \in U'$ and $U' \cap \mathcal{V}(P) \subseteq S$. We claim that

$$U'' = U \cup U' - \big(\mathcal{V}(P_1) \cup \cdots \cup \mathcal{V}(P_k)\big)$$

is the required neighborhood of P. It is evidently an open set that contains P. Suppose that $Q \in U''$. If $Q \in U$ then $Q \in S$. Otherwise, Q is in

$$X - U = \mathcal{V}(P) \cup \mathcal{V}(P_1) \cup \cdots \cup \mathcal{V}(P_k),$$

and this implies that $Q \in \mathcal{V}(P)$. But Q must also be in U', and $U' \cap \mathcal{V}(P) \subseteq S$. $\square$

We can use this to show:

Theorem. *Let R be an excellent ring. Then the Cohen-Macaulay locus*

$$\{P \in \operatorname{Spec}(R) : R_P \text{ is Cohen-Macaulay}\}$$

is Zariski open.

Proof. It suffices to estabish (1) and (2) of the preceding Lemma. We know (1) because if $P \subseteq Q$ then R_P is a localization of the Cohen-Macaulay ring R_Q. Now suppose that R_P is Cohen-Macaulay. Choose a maximal regular sequence in PR_P. After multiplying by suitable units in R_P, we may assume that this regular sequence consists of images of elements $x_1, \dots, x_d \in P$. We can choose $c_i \in R - P$ that kills the annihilator of x_{i+1} in