Math 711: Lecture of December 1, 2006

Recall the if R is a Noetherian ring of prime characteristic $p > 0$, R is called F-*finite* if $F : R \to R$ makes R into a module-finite algebra over

$$F(R) = \{r^p : r \in R\},$$

a subring of R that is also denoted R^p. When R is F-finite, the composition $F^e : R \to R$ also makes R into a finite module over

$$F^e(R) = \{r^{p^e} : r \in R\},$$

a subring of R that is alternatively denoted R^{p^e}.

If R is F-finite, it is trivial that every homomorphic image of R is F-finite. The same holds for each localization $W^{-1}R$, because inverting the elements in W^p has the effect of inverting the elements of W. If R is F-finite, so is $R[x]$: if $r_1, \ldots, r_h$ span R over $F(R)$, then the elements $r_i x_j$, $1 \le i \le h$, $0 \le j < p$ span $R[x]$ over $F(R[x]) = F(R)[x^p]$. By induction, any finitely generated algebra over an F-finite ring is F-finite, and it is likewise true that any algebra essentially of finite type over an F-finite ring is F-finite.

A perfect field is obviously F-finite, and so a field that is finitely generated as a field over a perfect field is F-finite: it is a localization of a finitely generated algebra over a perfect field. Thus, if K is perfect, each of the fields $K(t_1, \ldots, t_n)$ is F-finite, where $t_1, t_2, \ldots, t_n, \ldots$ are indeterminates over K, but the field $K(t_1, \ldots, t_n, \ldots)$ where we adjoin infinitely many indeterminates, is not. We note:

Proposition. *A complete local ring (R, m, K) of prime characteristic $p > 0$ is F-finite if and only if its residue class field K is F-finite.*

Proof. Since $K = R/m$, if R is F-finite then K is. Suppose that K is F-finite, and let $c_1, \ldots, c_h$ be a basis for K over $F(K)$. Then R is a homomorphic image of a formal power series ring $S = K[[x_1, \ldots, x_d]]$, and it suffices to show that S is F-finite. But the set of elements

$$\{c_j x_1^{a_1} \cdots x_d^{a_d} : 0 \le j \le h,\ 0 \le a_i < p \text{ for } 0 \le i \le d\}$$

spans S over $F(S) = F(K)[[x_1^p, \ldots, x_d^p]]$. $\square$

This justifies the assertion in the Lecture Notes of November 29 that a complete local ring with perfect residue class field is F-finite.

We next want to understand the behavior of the rank of ${}^eR$ when R is a complete local domain with a perfect residue class field.

Note that when R is reduced of prime characteristic $p > 0$, the three maps $F^e : R \to R$, $R^{p^e} \subseteq R$, and $R \subseteq R^{1/p^e}$ are isomorphic. The isomorphism of $F^e : R \to R$ with $R^{p^e} \subseteq R$

follows from the fact that, for a reduced ring R, F^e is injective and $F^e(R) = R^{p^e}$. To understand the third map, we need to define the ring R^{1/p^e}. When R is a domain, there we may take this to be the subring of an algebraic closure of the fraction field of R that consists of all the elements of the for r^{1/p^e} for $r \in R$. In the general case, one can show that there is an extension S of R, unique up to canonical isomorphism, such that the map $R = \{s^{p^e} : s \in S\}$. In fact, since $R \cong R^{p^e}$ via the map $r \mapsto r^{p^e}$, we "think of" R^{p^e} as R, and take S to be R.

This means that when R is reduced, we may think of ${}^eR$ as R^{1/p^e}.

Theorem. *Let (R, m, K) be a complete local ring of Krull dimension d such that K is perfect. Then for every $e \in \mathbb{N}$, the torsion-free rank of ${}^eR$ over R is p^{de}.*

Proof. By the structure theory of complete local rings, R is module finite over $A = K[[x_1, \ldots, x_d]]$. Let $\mathrm{frac}\,(R) = \mathcal{L}$ and $\mathrm{frac}\,(A) = \mathcal{K}$. The torsion free rank of R^{1/p^e} over R is the same as $[\mathcal{L}^{1/p^e} : \mathcal{L}]$. We have that

$$[\mathcal{L}^{1/p^e} : \mathcal{K}] = [\mathcal{L}^{1/p^e} : \mathcal{K}^{1/p^e}]\,[\mathcal{K}^{1/p^e} : \mathcal{K}]$$

and also

$$[\mathcal{L}^{1/p^e} : \mathcal{K}] = [\mathcal{L}^{1/p^e} : \mathcal{L}]\,[\mathcal{L} : \mathcal{K}],$$

so that

$$(*) \quad [\mathcal{L}^{1/p^e} : \mathcal{K}^{1/p^e}]\,[\mathcal{K}^{1/p^e} : \mathcal{K}] = [\mathcal{L}^{1/p^e} : \mathcal{L}]\,[\mathcal{L} : \mathcal{K}].$$

The map $u \to u^{1/p^e}$ gives an isomorphism of the inclusion $\mathcal{K} \subseteq \mathcal{L}$ with the inclusion $\mathcal{K}^{1/p^e} \subseteq \mathcal{L}^{1/p^e}$, so that

$$[\mathcal{L}^{1/p^e} : \mathcal{K}^{1/p^e}] = [\mathcal{L} : \mathcal{K}].$$

But then $(*)$ implies that

$$[\mathcal{L}^{1/p^e} : \mathcal{L}] = [\mathcal{K}^{1/p^e} : \mathcal{K}],$$

and the latter is the same as the torsion-free rank over A of

$$B = A^{1/p^e} \cong K[[x_1^{1/p^e}, \ldots, x_d^{1/p^e}]].$$

Let $y_i = x_i^{1/p^e}$, $1 \le i \le d$. Then B is free over A on the basis consisting of all monomials $y_1^{a_1} \cdots y_d^{a_d}$ with $0 \le a_i < p^e$ for $1 \le i \le d$. This free basis has cardinality $(p^e)^d = p^{de}$, as required. $\square$

We are now ready to prove the existence of Hilbert-Kunz multiplicities: the result is stated on the first page of the Lecture Notes of November 29, but we repeat the statement.

Theorem (Monsky). *Let M be a finitely generated module of dimension d over (R, m, K), where R has prime characteristic $p > 0$, and let $\mathfrak{A} \subseteq m$ be m-primary. Then the Hilbert-Kunz multiplicity $e_{HK}(\mathfrak{A}, M)$ of M with respect to $\mathfrak{A}$ exists, and is a positive real number.*

Proof. By the results of the Lecture of November 29, it suffices to prove this when $M = (R,\, m,\, K)$ is a complete local domain with a perfect residue class field. Let

$$\gamma_n = \frac{\ell(R/\mathfrak{A}^{[p^n]})}{p^{nd}}.$$

We shall prove that the sequence $\{\gamma_n\}_n$ is a Cauchy sequence. This will prove that the sequence has a limit. The fact that the limit is positive then follows from the lower bound in part (c) of the Lemma on p. 2 of the Lecture Notes of November 29.

The first key point is that ${}^1R \cong R^{1/p}$ has torsion-free rank p^d as an R-module. Thus, ${}^1R$ and $R^{\oplus p^d}$ become isomorphic after localization at a nonzero element of the domain R. By part (b) of the Lemma on p. 4 of the Lecture Notes of November 29, there is a positive real constant C such that

$$(*) \quad |\mathcal{F}_{HK}(\mathfrak{A},\, R^{\oplus p^d})(n) - \mathcal{F}_{HK}(\mathfrak{A},\, {}^1R)(n)| \le C/p^{(d-1)n}$$

for all $n \in \mathbb{N}$. The leftmost term is $p^d\mathcal{F}_{HK}(\mathfrak{A},\, R)(n)$. By the Proposition at the top of p. 4 of the Lecture Notes of November 29,

$$\mathcal{F}_{HK}(\mathfrak{A},\, {}^1R)(n) = \mathcal{F}_{HK}(\mathfrak{A},\, R)(n+1).$$

Thus, $(*)$ becomes

$$(**) \quad |p^d\mathcal{F}_{HK}(\mathfrak{A},\, R)(n) - \mathcal{F}_{HK}(\mathfrak{A},\, R)(n+1)| \le Cp^{(d-1)n}.$$

We may divide both sides by $p^{(n+1)d}$ to obtain

$$(***) \quad |\gamma_n - \gamma_{n+1}| \le C/p^{dn-n-dn-d} = \frac{Cp^{-d}}{p^n}.$$

Hence, for all $N \ge n$,

$$|\gamma_n - \gamma_N| \le |\gamma_n - \gamma_{n+1}| + |\gamma_{n+1} - \gamma_{n+2}| + \cdots + |\gamma_{N-1} - \gamma_N|$$
$$\le \frac{Cp^{-d}}{p^n}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \le \frac{Cp^{-d}(1-1/p)^{-1}}{p^n},$$

which shows that $\{\gamma_n\}_n$ is a Cauchy sequence, as claimed. □

The proof of the Theorem of Monsky can be easily adapted to show more.

Theorem. *Let $(R,\, m,\, K)$ by a local ring of prime characteristic $p > 0$, let M be a finitely generated R-module, and let $\mathfrak{A} \subseteq m$ be an m-primary ideal. Then the Hilbert-Kunz multiplicity with respect to $\mathfrak{A}$ is additive in the sense that if one has a finite filtration of M with factors N_i, $e_{HK}(\mathfrak{A},\, M)$ is the sum of the values of $e_{HK}(\mathfrak{A},\, N_i)$ for those N_i of the same dimension as M.*