

Math 711: Lecture of November 20, 2006

The following result can be deduced easily from the Buchsbaum-Eisenbud acyclicity criterion, but we give a short, self-contained argument.

Proposition. *Let (R, \mathfrak{m}, K) be a local ring and N a maximal Cohen-Macaulay module over R . If M is a finitely generated R -module of finite projective dimension over R , then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

Proof. For any prime ideal P in $\mathrm{Supp}(N)$, N_P is again a maximal Cohen-Macaulay module over R_P . (This is clear if $\mathrm{height}(P) = 0$. If $\mathrm{height}(P) > 0$, choose $x \in P$ not in any minimal prime of R . Then x is not a zerodivisor on N , and the result follows by Noetherian induction, since N/xN will be a maximal Cohen-Macaulay module for R/xR .) Let $\mathrm{pd}_R(M) = h$. Then $\mathrm{Tor}_i^R(M, N) = 0$ for $i > h$. If we have a counterexample, we can localize at a minimal prime P of $\bigoplus_{i=1}^h \mathrm{Tor}_i^R(M, N)$. Thus, we may assume without loss of generality that all of the non-vanishing $\mathrm{Tor}_i^R(M, N)$ for $i \geq 1$ have finite length, and we can choose i as large as possible for which one of these Tor modules is not 0. If

$$0 \rightarrow G_h \rightarrow \cdots \rightarrow G_0 \rightarrow 0$$

is a minimal free resolution of M over R , the modules $\mathrm{Tor}_i^R(M, N)$ are the homology modules of the complex

$$0 \rightarrow G_h \otimes_R N \rightarrow \cdots \rightarrow G_0 \otimes_R N \rightarrow 0.$$

Let d_j denote the map

$$G_j \otimes_R N \rightarrow G_{j-1} \otimes_R N.$$

Let $Z_j = \mathrm{Ker}(d_j)$ and let $B_j = \mathrm{Im}(d_{j+1})$. Thus, $Z_j = B_j$ for $j > i$. Note that we cannot have $\dim(R) = 0$, for then $\mathrm{pd}_R M \leq \mathrm{depth}_{\mathfrak{m}}(R) = 0$, and M is free. Thus, we may assume $\dim(R) \geq 1$. Also note that we cannot have $i = h$, because $\mathrm{Tor}_i^R(M, N) \subseteq G_h \otimes_R N$, a finite direct sum of copies of N , and has no submodule of finite length, since $\mathrm{depth}_{\mathfrak{m}} N = \dim(R) > 0$. Then we have a short exact sequence

$$0 \rightarrow B_i \rightarrow Z_i \rightarrow \mathrm{Tor}_i^R(M, N) \rightarrow 0$$

and an exact sequence

$$0 \rightarrow G_h \otimes_R N \rightarrow \cdots \rightarrow G_{i+1} \otimes_R N \rightarrow B_i \rightarrow 0.$$

Since $\mathrm{Tor}_i^R(M, N)$ is a nonzero module of depth 0, $Z_i \neq 0$, and $Z_i \subseteq G_i \otimes N$ has depth at least one. It follows that $\mathrm{depth}_{\mathfrak{m}}(B_i) = 1$. The exact sequences

$$0 \rightarrow B_j \rightarrow G_j \otimes_R N \rightarrow B_{j-1} \rightarrow 0$$

for $j > i$ enable us to see successively that $\mathrm{depth}_{\mathfrak{m}} B_{i+1} = 2$, $\mathrm{depth}_{\mathfrak{m}} B_{i+2} = 3$ and, eventually, $\mathrm{depth}_{\mathfrak{m}} B_{h-1} = (h-1) - (i-1) = h-i$. But $B_{h-1} = G_h \otimes_R N$ has depth $\dim(R) \geq \mathrm{depth}_{\mathfrak{m}} R \geq h > h-i$, since $i \geq 1$, a contradiction. \square

We next observe the following result related to the final Corollary of the Lecture of November 17.

Lemma. *Let (R, m, K) be local, and $f \in m^d - m^{d+1}$ be such that $\mathcal{L}(f)$ is part of a homogeneous system of parameters for $\text{gr}_m(R)$. Let N be a linear maximal Cohen-Macaulay module. Then $e(N/fN) = de(N)$.*

Proof. Without loss of generality we may replace R by $R(t)$ and N by $R(t) \otimes_R N$, and so assume that we have an infinite residue class field. Let $\dim(R) = r$, and let x_1, \dots, x_r be a minimal reduction of m . Then $\text{gr}_m(N) \cong (N/mN) \otimes_K K[X_1, \dots, X_r]$ is Cohen-Macaulay of depth r over $\text{gr}_m(R)$: the images of the x_j in m/m^2 form a regular sequence. It follows that $\mathcal{L}(f)$, which has degree d , is a nonzerodivisor on $\text{gr}_m(N)$, and we may apply part (b) of the final Corollary of the Lecture of November 17. \square

We are now ready to prove the result we are aiming for (cf. [J. Herzog, B. Ulrich, and J. Backelin, *Linear maximal Cohen-Macaulay modules over strict complete intersections*, Journal of Pure and Applied Algebra **71** (1991) 187–202.]

Theorem (Herzog, Ulrich, and Backelin). *Let (R, m, K) be a Cohen-Macaulay local ring that has a linear maximal Cohen-Macaulay module N . Let $f \in m^d - m^{d+1}$ be a nonzerodivisor such that its leading form $\mathcal{L}(f) \in \text{gr}_m(R)$ is part of a homogeneous system of parameters for $\text{gr}_m(R)$. Then R/fR has a linear maximal Cohen-Macaulay module.*

Proof. We adopt the notation of the Theorem on p. 3 of the Lecture of November 17, so that we have a chain

$$R^s = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_d = fR^s$$

as in the statement of that Theorem. Thus, the modules

$$0 \subseteq G_{d-1}/G_d \subseteq \cdots \subseteq G_i/G_d \subseteq \cdots \subseteq G_0/G_d$$

give an ascending filtration of $G_0/G_d \cong R^s/fR^s$ in which every factor module M_i is a maximal Cohen-Macaulay module over $\overline{R} = R/fR$ that is minimally generated by s elements and such that all of the modules G_i/G_d , G_0/G_i , and $M_i = G_i/G_{i+1}$ have finite projective dimension over R .

We shall show that the modules $G_i/G_d \otimes N$ give an ascending filtration of N^s/fN^s such that each factor $M_i \otimes_R N$ is a maximal Cohen-Macaulay module over \overline{R} . We then prove that at least one of $M_i \otimes_R N$ is a linear maximal Cohen-Macaulay module over \overline{R} .

First, we have short exact sequences

$$0 \rightarrow G_i/G_d \rightarrow G_0/G_d \rightarrow G_0/G_i \rightarrow 0,$$

$0 \leq i \leq d-1$ (this is the range for i throughout). These remain exact when we apply $-\otimes_R N$, since G_0/G_i has finite projective dimension over R and N is a maximal Cohen-Macaulay module: $\text{Tor}_1(G_0/G_i, N) = 0$. This shows that each $(G_i/G_d) \otimes_R N$ embeds in

$$G_0/G_d \otimes_R N \cong \overline{R}^s \otimes_R N \cong (N/fN)^{\oplus s}.$$

The short exact sequences

$$0 \rightarrow G_{i+1}/G_d \rightarrow G_i/G_d \rightarrow M_i \rightarrow 0$$

likewise remain exact when we apply $-\otimes_R N$, and so it follows that the factors are the modules $M_i \otimes_R N$.

We want to prove that each $M_i \otimes_R N$ is a maximal Cohen-Macaulay module over \bar{R} . We use the notation of the proof of the Theorem on p. 3 of the Lecture Notes of November 17. Recall that the complex

$$\dots \xrightarrow{\bar{\psi}'_i} \bar{R}^s \xrightarrow{\bar{\psi}_i} \bar{R}^s \xrightarrow{\bar{\psi}'_i} \bar{R}^s \xrightarrow{\bar{\psi}_i} \bar{R}^s \rightarrow 0$$

is acyclic, and that $\text{Coker } \psi_i = R^s/G_i$, which is also $\text{Coker } \bar{\psi}_i$. Moreover, we have that $\text{Im } \bar{\psi}_i \cong G_i/G_d$. Therefore we have short exact sequences:

$$0 \rightarrow R^s/G_i \rightarrow \bar{R}^s \rightarrow G_i/G_d \rightarrow 0 \quad \text{and} \quad 0 \rightarrow G_i/G_d \rightarrow \bar{R}^s \rightarrow R^s/G_i \rightarrow 0$$

for all i , $0 \leq i \leq d-1$. Since these modules have finite projective dimension over R , both sequences remain exact when we apply $-\otimes_R N$, yielding

$$(*) \quad 0 \rightarrow (R^s/G_i) \otimes_R N \rightarrow \bar{N}^s \rightarrow (G_i/G_d) \otimes_R N \rightarrow 0$$

and

$$(**) \quad 0 \rightarrow (G_i/G_d) \otimes_R N \rightarrow \bar{N}^s \rightarrow (R^s/G_i) \otimes_R N \rightarrow 0.$$

If

$$k = \text{depth}_m((G_i/G_d) \otimes_R N) < \dim(\bar{R}) - \text{depth}_m \bar{N},$$

then (*) shows that

$$\text{depth}_m((R^s/G_i) \otimes_R N) = k + 1,$$

and then (**) shows that

$$\text{depth}_m((G_i/G_d) \otimes_R N) \geq k + 1 > k,$$

a contradiction. If

$$\text{depth}_m((R^s/G_i) \otimes_R N) < \dim(\bar{R}) - 1,$$

we get an entirely similar contradiction by first using (**) and then (*).

The exact sequences

$$0 \rightarrow M_i \rightarrow R^s/G_{i+1} \rightarrow R^s/G_i \rightarrow 0$$

likewise remain exact when we apply $-\otimes_R N$, yielding exact sequences

$$0 \rightarrow M_i \otimes_R N \rightarrow (R^s/G_{i+1}) \otimes_R N \rightarrow (R^s/G_i) \otimes_R N \rightarrow 0.$$