Math 711: Lecture of October 7, 2005

For the purpose of the next theorem, we make the convention that the type of the 0 module over a local ring R is ≤ 1. (It should be the vector space dimension of $\mathrm{Ext}_R^{-1}(K,\, M) = 0$, i.e., it should be 0.)

Theorem. *Let R be a homomorphic image of a regular ring, and let M be a finitely generated R-module. Let $t \geq 1$ be a fixed integer. Then the set*

$$\{P \in \mathrm{Spec}\,(R) : M_P \text{ is Cohen-Macaulay of type } \leq t\}$$

is Zariski open in $\mathrm{Spec}\,(R)$.

Proof. Let $R = S/J$, where S is regular. Then $\mathrm{Spec}\,(R)$ is homeomorphic with the closed set $V(J) \subseteq \mathrm{Spec}\,(S)$: if we identify $\mathrm{Spec}\,(R)$ with $V(J)$, the locus we want in $\mathrm{Spec}\,(R)$ is the locus for $\mathrm{Spec}\,(S)$ intersected with $V(J)$. Thus, it suffices to consider the problem for S instead, and we may assume without loss of generality that R is regular.

Let P be a prime of R such that M_P is Cohen-Macaulay of type at most t. If M_P is 0, this will be true on a Zariski neighborhood of P, and we assume $M_P \neq 0$. By the preceding result, we may localize at one element of $R - P$ so that M will be Cohen-Macaulay with annihilator of pure height h. Then $\mathrm{Ext}_R^h(M,\, R)_P \cong \mathrm{Ext}_{R_P}^h(M_P,\, R_P) = M_P^*$ can be generated by t or fewer elements, and by clearing denominators we may assume that these elements have the form $u_1/1,\, \ldots\,,\, u_t/1$ where $u_i \in \mathrm{Ext}_R^h(M,\, R)$ for all i. (If fewer than t generators are needed we may take some of the u_i to be 0.) Let N be the R-span of the u_i. Then $\big(\mathrm{Ext}_R^h(M,\, R)/N\big)_P = 0$ and so we can localize at one element of $R - P$ that kills $\mathrm{Ext}_R^h(M,\, R)/N$. After this localization, we have that $\mathrm{Ext}_R^h(M,\, R) = N$ is generated by at most t elements, and so for all Q,

$$\mathrm{Ext}_R^h M,\, R)_Q \cong \mathrm{Ext}_{R_Q}^h(M_Q,\, R_Q) = M_Q^*$$

has at most t generators. But this implies that the type of M_Q is at most t, as required. $\square$

Corollary. *Let R be a homomorphic image of a regular ring. Then*

$$\{P \in \mathrm{Spec}\,(R) : R_P \text{ is Gorenstein}\}$$

is Zariski open in $\mathrm{Spec}\,(R)$.

Proof. We may apply the preceding result with $M = R$. The fact that the type of R_P is at most one implies that it is exactly one. $\square$

We have already proved for a local flat homomorphism $(R,\, m,\, K) \to (S,\, n,\, L)$ of local rings that S is Cohen-Macaulay (respectively, Gorenstein) if and only if both R and S/mS are Gorenstein. We next want to give a global version of this result that also describes the behavior of the loci where these properties fail. We treat the Cohen-Macaulay and Gorenstein cases simultaneously by axiomatizing the properties we need.

Recall that if $R \to S$ is a ring homorphism, its fiber over $P \in \mathrm{Spec}\,(R)$ is $\kappa_P \otimes_R S$, where $\kappa_P = R_P/PR_P \cong \mathrm{frac}\,(R/P)$, the fraction field of R/P. Thus, the fiber may also be described as $(R-P)^{-1}S/PS$. The map $S \to (R-P)^{-1}S/PS$ induces an injection $\mathrm{Spec}\,\big((R-P)^{-1}S/PS\big) \hookrightarrow \mathrm{Spec}\,(S)$ whose image is the set of prime ideals of S lying over P in R. Thus, the primes in the spectrum of the fiber are in bijective correspondence with the prime ideals of S that contract to P.

Theorem. *Let $\mathcal{P}$ denote a property of Noetherian rings such that:*
(1) *If a local ring R has $\mathcal{P}$, so does its localization at any prime.*
(2) *R has $\mathcal{P}$ if and only if its localization at every maximal ideal has $\mathcal{P}$ (it then follows that all of its localizations have $\mathcal{P}$).*
(3) *If $(R, m, K) \to (S, n, L)$ is a local map of local rings, then S has $\mathcal{P}$ if and only if R has $\mathcal{P}$.*

Then the following statements hold:
(a) *If S is faithfully flat over R, then S has $\mathcal{P}$ if and only if R has $\mathcal{P}$ and every fiber of $R \to S$ has $\mathcal{P}$.*
(b) *If $R \to S$ is flat, all of the fibers have $\mathcal{P}$, and I is an ideal of S such that $V(I)$ is the set of primes of R that do not have property $\mathcal{P}$, then $V(IS)$ is the set of primes of S that do not have property $\mathcal{P}$.*

In particular, these results hold when $\mathcal{P}$ is the property of being Cohen-Macaulay and when $\mathcal{P}$ is the property of being Gorenstein.

Proof. We first consider part (a). Assume that S has $\mathcal{P}$. For every prime P of R there is a prime Q of S lying over P. Since S_Q has $\mathcal{P}$, so does R_P. Therefore, S has $\mathcal{P}$ implies that R has $\mathcal{P}$. Each prime of the fiber $(R-P)^{-1}(S/PS)$ corresponds to a prime Q of S lying over P, and it suffices to show that every ring $\big((R-P)^{-1}(S/PS)\big)_Q$ has $\mathcal{P}$. But this ring $\cong S_Q/PS_Q$, which has $\mathcal{P}$ because S does.

Now assume that R and all fibers have $\mathcal{P}$. Let Q be a prime of S lying over P in R. It suffices to show that S_Q has P. This is true because R_P and S_Q/PS_Q both have $\mathcal{P}$: the latter is a localization of $(R-P)^{-1}S/PS$.

To prove (b), let Q be a prime ideal of S and let P be its contraction to R. Note that $Q \in V(IS) \Leftrightarrow P \in V(I)$. If S_Q has $\mathcal{P}$, so does R_P, and so $P \notin V(I)$ and $Q \notin V(IS)$. If $Q \in V(IS)$ then $P \in V(I)$, so that R_P does not have $\mathcal{P}$ and S_Q does not have $\mathcal{P}$. □

Note that when R is a Hodge algebra over K on H governed by Σ, arbitrary base change on K produces a new Hodge algebra with the same data. More precisely, if $K \to K'$ is any ring homomorphism, $R' = K' \otimes_K R$, H' is the image of H in R' under the map sending $h \mapsto 1 \otimes h$, and Σ' is the semigroup corresponding to Σ under the obvious isomorphism $\mathbb{N}^H \to \mathbb{N}^{H'}$, then R' is a Hodge algebra over K' on H' governed by Σ'. The free basis of standard monomials for R evidently maps bijectively to a free basis for R' over K', and the straightening relations for R map to the required straightening relations for R'. In particular, each fiber $\kappa_P \otimes_K R$ is a Hodge algebra over a field.

Corollary. *A Hodge algebra over a Noetherian ring K is Cohen-Macaulay (respectively, Gorrenstein) if and only if K is Cohen-Macaulay (respectively, Gorenstein) and each fiber* *is Cohen-Macaulay (respectively, Gorenstein). The same holds for any property of rings $\mathcal{P}$ satisfying the three conditions in the Theorem above.*

The condition that each fiber is Cohen-Macaulay (respectively, Gorenstein) is equivalent to the condition that for every field κ to which K maps, $\kappa \otimes_K R$ is Cohen-Macaulay (respectively, Gorenstein).

Proof. The Hodge algebra is a free over K on a basis containing 1, and is therefore faithfully flat over K. The result is immediate from part (a) of the Theorem just above. The final statement follows from the fact that if P is the kernel of $K \to \kappa$ the map to κ factors through the fiber K_P/PK_P. The final statement now follows from the Lemma just following. $\square$

Lemma. *Let B be a finitely generated κ-algebra. Then B is Cohen-Macaulay (respectively, Gorenstein) if and only if $B' = \kappa' \otimes_\kappa B$ has the specified property for every field extension κ' of κ.*

Proof. B' is faithfully flat over B and so the "if" part follows. Now assume that B has the specified property. The result will follow if each fiber is Gorenstein (the fibers are then Cohen-Macaulay as well). Each fiber has the form $\kappa' \otimes_\kappa L$ where L has the form B_P/PB_P and so is a field finitely generated over κ'. We proceed by induction on the number of generators of the field L over κ. If $\kappa \subseteq L_0 \subseteq L$, we have that

$$\kappa' \otimes_\kappa L \cong (\kappa' \otimes_\kappa L_0) \otimes_{L_0} L.$$

Therefore, it suffices to show that if C is a Gorenstein algebra containing a field L_0 and L is s field generated over L_0 by one element, then $D = C \otimes_{L_0} L$ is Gorenstein. There are two cases. If $L = L_0(x)$ where x is transcendental over L_0, then $C \otimes_{L_0} L$ is a localization of $C[x]$, and this is Gorenstein, since it is flat over C with Gorenstein fibers. If L is generated by one element θ over L_0 and is algebraic, let f be the minimal monic polynomial of θ over L_0. Then $D \cong C \otimes_{L_0} L_0[x]/(f) \cong C[x]/(f)$. But $C[x]$ is Gorenstein, and the monic polynomial f is a nonzerodivisor. Thus, the quotient is also Gorenstein. $\square$

The following theorem gives that the defining radical ideal of the closed set of primes where a graded ring is not Cohen-Macaulay or not Gorenstein is homogeneous. We need a preliminary fact.

Lemma. *If T is flat over a reduced ring R and the fibers are reduced then T is reduced.*

Proof. If T has a nilpotent element other than 0, we may localize at a minimal prime Q of its annihilator, and if Q lies over P we may study $R_P \to T_Q$ instead. Then T_Q has depth 0, and so R_P has depth 0. Since this ring is reduced and local, it must be a field. But then T_Q is the fiber over P (a localization of the original fiber over P) and is reduced. $\square$

Theorem. *Let $S \subseteq \mathbb{N}^h$ be a semigroup and let R be a Noetherian ring graded by S. Suppose that the set of primes such that R_P is not Cohen-Macaulay (respectively, not Gorenstein) is closed. Then the radical ideal I defining this locus is homogeneous in the S-grading.*

Proof. There is no loss of generality in assuming that $S = \mathbb{N}^h$: we can enlarge S, and define the new graded pieces to be 0. For each i, $1 \le i \le h$, we can put a $\mathbb{Z}$-grading on