Math 711: Lecture of December 16, 2005

We continue the notations of last time, so that $R = K[u, v, x, y]_m/(uy - xv)$ and $P = (u, v)R$, a height one prime in R, which has dimension 3. We next want to determine the minimal resolution of R/P over R.

Proposition. *With notation as above, a minimal resolution for R/P over R is:*

$$\cdots \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R^2 \xrightarrow{\alpha} R^2 \xrightarrow{\beta} R^2 \xrightarrow{\alpha} R^2 \xrightarrow{(u\ v)} R \to R/P \to 0$$

where $\alpha = \begin{pmatrix} y & -v \\ -x & u \end{pmatrix}$ *and* $\beta = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$.

Proof. We first want to show that the columns of α, which give relations on u and v, span the module of all such relations. Suppose we lift the relation to the localized polynomial ring, so that we have $cu + dv = f(uy - vx)$. In the localized polynomial ring, we have that $cu \in (y, v)$, so that $c \in (y, v)$. It follows that by subtracting a linear combination of the columns of α we obtain a relation such that $c = 0$, i.e., we have $dv = f(uy - vx)$. But then the prime element $uy - vx$ divides d, so that $d = 0$ in R as well.

We next want to show that the columns of β, which are relations on the columns of α, give a basis for the relations on the columns of α. Since the determinant of α is 0 and R is a domain, giving a relation on the columns of α is the same as giving a relation on the initial entries y and $-v$. In fact, the projection map on the first coordinate gives an isomorphism of the span of the columns with the ideal $(y, -v)$, since the map is a surjection of torsion-free modules of rank one and so must be injective as well. Again, we lift the relation to the localized polynomial ring, obtaining $cy - dv = f(uy - vx)$. It follows that $cy \in (u, v)$, and so $c \in (u, v)$. It follows that by subtracting a linear combination of the columns of β, we get a relation such that $c = 0$, and the argument can be completed as before.

Again, since the determinant of β is 0, giving a relation on the columns is the same as giving a relation on the initial entries u and v. This is the problem we solved in the first paragraph, and we can now see that the resolution will be periodic with period two from this point on. □

Corollary. *Let R and P be as above, and let $Q = (y, v)$. Then there are short exact sequences*

$$0 \to Q \to R^2 \to P \to 0$$

and

$$0 \to P \to R^2 \to Q \to 0,$$

so that $Q = \mathrm{syz}^1 P$ and $P = \mathrm{syz}^1 Q$. Both P and Q are maximal Cohen-Macaulay modules over R, i.e., have depth 3.

Proof. Because each of the matrices α, β has determinant 0, projection onto the first coordinates gives isomorphisms of the column space of α with Q and of the column space of β with P. When a module over a Cohen-Macaulay ring has depth $\delta < d$, the dimension of the ring, its first module of syzygies has depth $\delta + 1$. Thus, P and Q, each of which is an Nth syzygy for all N, must have depth 3. □

Proposition. *We continue the notations of the preceding Proposition. Let M be an R-module of finite length n, so that M is an n-dimensional vector space over the field K. Let A, B, C, and D be the matrices of the actions of u, v, x, and y on M. Then A, B, C, D are mutually commuting nilpotent matrices such that $AD = BC$. Conversely, any four commuting nilpotent $n \times n$ matrices A, B, C, and D such that $AD = BC$ determine a length n module over R on which the actions of u, v, x, and y are those of the respective matrices.*

Let α^* denote the $2n \times 2n$ matrix over K whose block form is $\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$ and call the rank of this matrix ρ. Let β^* denote the rank of the $2n \times 2n$ matrix over K whose block form is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and call the rank of this matrix σ. The module M has finite projective dimension if and only if $\rho + \sigma = 2n$.

In terms of these ranks, $\chi(M,\, R/P) = n - \sigma$.

Proof. The statements in the first paragraph are clear. We know that M has finite projective dimension if and only if the complex $G_\bullet$ that resolves R/P is such that $G_\bullet \otimes_R M$ is exact at the jth spot for $j \gg 0$. But this complex eventually has the periodic form:

$$\cdots \xrightarrow{\alpha^*} M \oplus M \xrightarrow{\beta^*} M \oplus M \xrightarrow{\alpha^*} M \oplus M \xrightarrow{\beta^*} \cdots$$

and the condition for exactness in the case of a complex of finite-dimensional K vector spaces is that the rank of each incoming map is equal to the nullity of the outgoing map, which is the same as the condition that for each vector space, the sum of the ranks of the incoming and outgoing maps is the dimension of the vector space. At every spot, this condition is simply that $\rho + \sigma = 2n$.

For a maximal Cohen-Macaulay C over R, $\mathrm{Tor}_i^R(M,\, C) = 0$ for $i \geq 1$ when M has finite length and finite projective dimension. Hence $\chi(M,\, C) = \ell(M \otimes_R C)$. Then

$$\chi(M,\, R/P) = \chi(M,\, R) - \chi(M,\, P) = \ell(M \otimes_R R) - \ell(M \otimes_R P) =$$

$$\ell(M) - \ell(M \otimes_R \operatorname{Coker} \alpha) = n - \ell(\operatorname{Coker} \alpha^*) = n - (2n - \rho) = n - \sigma,$$

as claimed. □

Thus, the problem comes down to exhibiting $n \times n$ commuting nilpotent matrices A, B, C, and D such that $AD = BC$ and $\sigma + \rho = 2n$ but $\sigma = n + 1$. This is possible over any field when $n = 15$ (and not for any smaller n, by a result of Marc Levine.)

Below are the matrices that give an example where the intersection multiplicity is negative, in block form. The rows are grouped in blocks of respective sizes 2, 3, 2, 2, and 6, while the columns are grouped in blocks of respective sizes 5, 2, 2, 2, 2, and 2. First, let

$$E_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ F_{3\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},\ G_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},\ \text{and } H_{3\times 2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We use $0_r$and 1_r for the $r \times r$ zero matrix and the $r \times r$ identity matrix, respectively, and we use $0_{r\times s}$ for the $r \times s$ zero matrix. Let A, B, C, and D be the respective 15×15 matrices

$$\begin{pmatrix} 0_{2\times 5} & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_{3\times 5} & 0_{3\times 2} & 0_{3\times 2} & 0_{3\times 2} & 0_{3\times 2} & E_{3\times 2} \\ 0_{2\times 5} & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 \\ 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 \\ 0_{6\times 5} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} \end{pmatrix},\ \begin{pmatrix} 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_{3\times 5} & 0_{3\times 2} & 0_{3\times 2} & F_{3\times 2} & G_{3\times 2} & 0_{3\times 2} \\ 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_{6\times 5} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} \end{pmatrix},$$

$$\begin{pmatrix} 0_{2\times 5} & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 \\ 0_{3\times 5} & 0_{3\times 2} & 0_{3\times 2} & 0_{3\times 2} & 0_{3\times 2} & H_{3\times 2} \\ 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 \\ 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 \\ 0_{6\times 5} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} \end{pmatrix},\ \text{and}\ \begin{pmatrix} 0_{2\times 5} & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 \\ 0_{3\times 5} & 0_{3\times 2} & 0_{3\times 2} & 0_{3\times 2} & H_{3\times 2} & E_{3\times 2} \\ 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_{2\times 5} & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_{6\times 5} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} & 0_{6\times 2} \end{pmatrix}$$

Then we have that A, B, C, and D mutually commute, that $AD = BC$, and that $\rho = 14$, and $\sigma = 16$. Thus, the corresponding module M has finite projective dimension and $\chi(M, R/P) = -1$. Note that it is clear that these matrices are nilpotent, since they are upper triangular with all diagonal entries equal to zero. It is also easy to check that $m^3 + (v, y)m$ kills M. The detailed verification of all of these statements, which involves only routine calculations with matrices, is given in [S. P. Dutta, M. Hochster, and J. E. McLaughlin, *Modules of finite projective dimension with negative intersection multiplicities*, Invent. Math. **79** (1985) 253–291], where it is also shown that the minimal free resolution of this module has the form

$$0 \to R^5 \to R^16 \to R^{17} \to R^6 \to M \to 0.$$

One may consider the Grothendieck group of modules of finite length and finite projective dimension over R. Marc Levine has shown that this Grothendieck group is generated by the classes of the form $R/(x_1, x_2, x_3)R$ where x_1, x_2, x_3 is a maximal regular sequence, together with the class of the module M constructed above, and that there is no module M' of length smaller than 15 of finite projective dimension such that $\chi(M', R/P) \neq 0$. Cf. [M. Levine, *Localization on singular varieties*, Invent. Math. **91** (1988) 423–464] and [M. Levine, *Erratum to "Localization on singular varieties"*, Invent. Math. **93** (1988) 715–716]. Also see [P. C. Roberts and V. Srinivas, *Modules of finite length and finite projective dimension*, Invent. Math. **151** (2003) 1–27].

Let us called a finitely generated module M of finite projective dimension over a local ring R *descendable* if there is a flat local homomorphism $A \to R$, and a finitely generated A-module M_0 such that $M = R \otimes_A M_0$. We note that when M is descendable and N is finitely generated such that $M \otimes_R N$ has finite length and $\dim(M) + \dim(N) < \dim(R)$, then $\chi(M, N) = 0$. To see this, first note that the issues are unaffected by completing A,