Math 711: Lecture of September 29, 2006

We prove the Corollary stated at the end of the Lecture of September 27.

Proof. If the ideals are different, we may localize at a prime of R_2 in the support of the module which is their sum modulo their intersection. Thus, we may assume without loss of generality that R_2 is local. We may replace R_1 by its localization at the contraction of the maximal ideal of R_2, and R_0 by its localization at the contraction of the maximal ideal of R_2 (or of R_1: the contractions are the same). Thus, we may assume that we are in the case where $R_0 \hookrightarrow R_1 \hookrightarrow R_2$ are local and the homomorphisms are local. By the Lemma near the bottom of the page 4 of the Lecture Notes of September 27, we have that

$$R_1 \cong R_0[X_1, \ldots, X_m]_{\mathcal{P}}/(F_1, \ldots, F_m)$$

for some m and prime $\mathcal{P}$ of $R_0[X_1, \ldots, X_m]$. Then $\mathcal{J}_{R_1/R_0}$ is generated by the image of $\dfrac{\partial(F_1, \ldots, F_m)}{\partial(X_1, \ldots, X_m)}$. Likewise,

$$R_2 \cong (R_1[Y_1, \ldots, Y_s])_{\mathcal{Q}}/(G_1, \ldots, G_s).$$

Again, $\mathcal{J}_{R_2/R_1}$ is generated by the image of $\dfrac{\partial(G_1, \ldots, G_s)}{\partial(Y_1, \ldots, Y_s)}$. It follows that we can write R_2 as a localization of

$$R_0[X_1, \ldots, X_m, Y_1, \ldots, Y_s]/(F_1, \ldots, F_m, G_1, \ldots, G_s),$$

where the F_j do not involve the Y_i. This means that the Jacobian matrix has the block form

$$\begin{pmatrix} M_F & N \\ 0 & M_G \end{pmatrix}$$

where M_F is $\big(\partial F_j/\partial X_i\big)$ and M_G is $\big(\partial G_j/\partial Y_i\big)$. We have that $\mathcal{J}_{R_2/R_0}$ is generated by the image of the determinant of this matrix. No matter what N is, this determinant is

$$\det(M_F)\det(M_G) = \frac{\partial(F_1, \ldots, F_m)}{\partial(X_1, \ldots, X_m)}\,\frac{\partial(G_1, \ldots, G_s)}{\partial(Y_1, \ldots, Y_s)}$$

as required. $\square$

There are now three results whose proof are hanging: one is the proof that S' is module-finite over S, the second is the proof of the Key Lemma, which is stated on p. 3 of the Lecture Notes of September 27, and the third is the proof of the Jacobian Theorem itself. We begin with the proof of the Key Lemma. This will involve studying quadratic transforms of a regular local ring along a valuation. We first indicate our approach to the proof of the Key Lemma.

Proof of the Key Lemma: step 1. We are trying to show that $\mathcal{J}_{S'/R} \subseteq vS'$. Assume the contrary. Consider the primary decomposition of vS'. Since we are assuming that S' is module-finite over S (we still need to prove this), S' is a normal Noetherian ring, and the associated primes of vS' have height one. We may choose such a prime Q such that $\mathcal{J}_{S'/R}$ is not contained in the corresponding primary ideal in the primary decomposition of vS'. Since the elements of $S' - Q$ are not zerodivisors on this primary ideal, we also have that $\mathcal{J}_{S'_Q/R}$ is not contained in $S'_Q = V$. Thus, for the purpose of proving the Key Lemma, we may replace S' by V, which is a discrete valuation ring, and we may replace R by its localization at the contraction of Q to R. In the remainder of the argument we may therefore assume[1] that (R, m, K) is regular local.

We now digress to discuss quadratic transforms. We first want to prove:

Lemma. *Let (R, m, K) be regular local with regular system of parameters $x_1, \dots, x_d$. Then $T = R[x_2/x_1, \dots, x_d/x_1] = R[m/x_1]$ is regular, and the images of $x_2, \dots, x_d$ are algebraically independent over K in $T/x_1T \cong K[\overline{x}_2, \dots, \overline{x}_d]$.*

Proof. If we localize T at a prime that does not contain x_1, the resulting ring is a localization of R_{x_1}, and is therefore regular. For primes that contain x_1, it suffices to show that the localization is regular after killing x_1, and this follows from the fact that T/x_1T is regular even without localizing. It therefore suffices to prove that T/x_1T is a polynomial ring.

First note that

$$T = \bigcup_k m^k/x_1^k,$$

where $m^k/x_1^k = \{u/x_1^k : u \in m^k\}$, and this is an increasing union since $m^k/x_1^k = x_1m^k/x_1^{k+1}$. (It is clear that $m^k/x_1^k \subseteq T$, and the product of the j th and k th terms in the union is the $(j+k)$ th term.) Hence, if there is a relation among the $\overline{x}_j$, $2 \le j \le n$, we may lift it obtain a nonzero polynomial g whose nonzero coefficients are units of R (we get them by lifting elements of K to R) such that $g(x_2/x_1, \dots, x_d/x_1) = x_1t$ where $t \in m^k/x_1^k$. We multiply both sides by x_1^N where $N \ge k$ is also larger than the absolute value of any negative exponent on x_1 occurring on the left. The left hand side becomes a nonzero homogeneous polynomial G of degree N in $x_1, \dots, x_d$ whose nonzero coefficients are units. The right hand side is in $x_1 \cdot x_1^{N-k}m^k \subseteq x_1^{1+N-k}m^k \subseteq m^{N+1}$. This gives a nonzero relation of degree N on the images of $x_1, \dots, x_d$ in $\mathrm{gr}_m(R)$, a contradiction, since when R is regular this is a polynomial ring in the images of the x_j. □

Definition. Let (R, m, K) be a regular local ring of Krull dimension $d \ge 2$ and let suppose that $R \subseteq V$ is a local map to discrete valuation ring V. Let $\mathrm{ord}_V = \mathrm{ord}$ denote

[1] Note that in the refined version of the Jacobian theorem, it was assumed that R_P is regular if P lies under a height one prime of S', so that we may make this reduction even in that case.

the corresponding valuation. By the *immediate* or *first quadratic transform* of R *along* V we mean the following: let $x_1, \ldots, x_d$ be a regular system of parameters for R numbered so that $\operatorname{ord}(x_1) \leq \operatorname{ord}(x_j)$ for $j \geq 2$, let $T = R[x_2/x_1, \ldots, x_d/x_1]$, and then the first quadratic transform is T_P, where P is the contraction to T of the maximal ideal of V. Then T_P is again regular, and we have a local map $T_P \subseteq V$. We may therefore iterate to obtain a sequnce of quadratic transforms of R, called the *quadratic sequence* of R *along* V. The sequence is finite if it eventually contains a ring of dimension 1. We are aiming to prove that the sequence is finite whenever the transcdence degree of the residue class field of V is of $d-1$, in which case the ring $V \cap \operatorname{frac}(R)$ occurs as the final term.