

Math 711: Lecture of September 14, 2007

The following result is very useful in thinking about tight closure.

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$, let $N \subseteq M$ be R -modules, and let $u \in M$. Then $u \in N_M^*$ if and only if the image \bar{u} of u in the quotient M/N is in $0_{M/N}^*$.*

Hence, if we map a free module G onto M , say $h : G \twoheadrightarrow M$, let $H = h^{-1}(N) \subseteq G$, and let $v \in G$ be such that $h(v) = u$, then $u \in N_M^$ if and only if $v \in H_G^*$.*

Proof. For the first part, let $c \in R^0$. Note that, by the right exactness of tensor products, $\mathcal{F}^c(M/N) \cong \mathcal{F}^c(M)/N^{[q]}$. Consequently, $cu^q \in N^{[q]}$ for all $q \geq q_0$ if and only if $c\bar{u}^q = 0$ in $\mathcal{F}^c(M/N)$ for $q \geq q_0$.

For the second part, simply note that the image of v in $G/H \cong M/N$ corresponds to \bar{u} in M/N . \square

It follows many questions about tight closure can be formulated in terms of the behavior of tight closures of submodules of free modules. Of course, when M is finitely generated, the free module G can be taken to be finitely generated with the same number of generators.

Given a free module G of rank n , we can choose an ordered free basis for G . This is equivalent to choosing an isomorphism $G \cong R^n = R \oplus \cdots \oplus R$. In the case of R^n , one may understand the action of Frobenius in a very down-to-earth way. We may identify $\mathcal{F}^c(R^n) \cong R^n$, since we have this identification when $n = 1$. Keep in mind, however, that the identification of $\mathcal{F}^c(G)$ with G depends on the choice of an ordered free basis for G . If $u = r_1 \oplus \cdots \oplus r_n \in R^n$, then $u^q = r_1^q \oplus \cdots \oplus r_n^q$. With $H \subseteq R^n$, $H^{[q]}$ is the R -span of the elements u^q for $u \in H$ (or for u running through generators of H). Very similar remarks apply to the case of an infinitely generated free module G with a specified basis b_λ . The elements b_λ^q give a free basis for $\mathcal{F}^c(G)$, and if $u = r_1 b_{\lambda_1} + \cdots + r_s b_{\lambda_s}$, then $u^q = r_1^q b_{\lambda_1}^q + \cdots + r_s^q b_{\lambda_s}^q$ gives the representation of u^q as a linear combination of elements of the free basis $\{b_\lambda^q\}_\lambda$.

We could have defined tight closure for submodules of free modules using this very concrete description of u^q and $H^{[q]}$. The similarity to the case of ideals in the ring is visibly very great. But we are then saddled with the problem of proving that the notion is independent of the choice of free basis. Moreover, if we take this approach, we need to define N_M^* by mapping a free module G onto M and replacing N by its inverse image in G . We then have the problem of proving that the notion we get is independent of the choices we make.

Our next objective is to prove that in a regular ring, every ideal is tightly closed. This depends on knowing that $F : R \rightarrow R$ is flat for regular rings of prime characteristic $p > 0$.

Eventually we sketch below a proof of the flatness of F that depends on the structure theory for complete local rings of prime characteristic $p > 0$. Later, we shall give a different proof, based on the following result, which is valid without restriction on the characteristic:

Theorem. *Let (R, m, K) be a regular local ring and M an R -module. Then M is a big Cohen-Macaulay module for R if and only if M is faithfully flat over R .*

We postpone the proof of this result for a while: it makes considerable use of the properties of the functor Tor . However, we do want to make several comments.

First note that it immediately implies that when R is regular, $F : R \rightarrow R$ is flat. In general, $R \rightarrow S$ is flat if and only if for every prime ideal Q of S with contraction P to R , the map $R_P \rightarrow S_Q$ is flat.

To see this, note that for R_P -modules M , the natural map $S_Q \otimes_R M \rightarrow S_Q \otimes_{R_P} M$ is an isomorphism, because $M \rightarrow R_P \otimes_{R_P} M$ is an isomorphism, and we have

$$S_Q \otimes_{R_P} M \cong S_Q \otimes_{R_P} (R_P \otimes_R M) \cong S_Q \otimes_R M.$$

The latter is also $S_Q \otimes_S (S \otimes_R M)$. If S is flat over R , since S_P is flat over S we have that S_P is flat over R . On the other hand, if $N \hookrightarrow M$ is an injection of R -modules and $S \otimes_R N \rightarrow S \otimes_R M$ is not injective, we can localize at a prime Q of S in the support of the kernel. This yields a map $S_Q \otimes_R N \rightarrow S_Q \otimes_R M$ that is not injective. But if Q contracts to P , we do have that $N_P \rightarrow M_P$ is injective. This shows that S_Q is not flat over R_P . \square

Note that when $S = R$ and the map is F , the contraction of $P \in \text{Spec}(R)$ is P . Thus, it suffices to show that F is flat on R_P for all primes P . This is now obvious given the Theorem above: any regular sequence (equivalently, system of parameters) in R , say x_1, \dots, x_n , maps to x_1^p, \dots, x_n^p in R , which is again a regular sequence. Hence, R is a big Cohen-Macaulay algebra for R under the map $F : R \rightarrow R$, and this proves that R is faithfully flat over R .

We have the following additional comments on the Theorem. Suppose that M is a module over a local ring (R, m, K) and suppose that we know that x_1, \dots, x_n is a system of parameters that is a regular sequence on M . Let $\mathfrak{A} = (x_1, \dots, x_n)R$. By the definition of a regular sequence, we have that $\mathfrak{A}M \neq M$. We want to point out that this condition implies the *a priori* stronger condition that $mM \neq M$. The reason is that m is nilpotent modulo \mathfrak{A} . Thus, we can choose s such that $m^s \subseteq \mathfrak{A}$. If $M = mM$, we can multiply by m^t to conclude that $m^t M = m^t(mM) = m^{t+1}M$. Thus

$$M = mM = m^2M = \dots = m^t M = \dots.$$

Then

$$M = m^s M \subseteq \mathfrak{A}M \subseteq M,$$

and we find that $\mathfrak{A}M = M$, a contradiction.

If M is faithfully flat over R , we have that $(R/m) \otimes M = M/mM \neq 0$, so that $mM \neq M$. Moreover, whenever x_1, \dots, x_n is a system of parameters for R , it is a regular sequence on R , and the fact that M is faithfully flat over R implies that x_1, \dots, x_n is a regular sequence on M . This shows that a faithfully flat R -module is a big Cohen-Macaulay module over R . The converse remains to be proved.

We next sketch a completely different proof that F is flat for a regular ring R . As noted above, this comes down to the local case. We use the fact that a local map $R \rightarrow S$ of local rings is flat if and only if the induced map $\widehat{R} \rightarrow \widehat{S}$ is flat. Hence, by the structure theory of complete local rings, we may assume that $R = K[[x_1, \dots, x_n]]$ is a formal power series ring over a field. Since this ring is the completion of $K[x_1, \dots, x_n]_P$ where $P = (x_1, \dots, x_n)$, it suffices to prove the result for the localized polynomial ring $R = K[x_1, \dots, x_n]$ itself. But $F(R) = K^p[x_1^p, \dots, x_n^p]$. Thus, all we need to show is that $K^p[x_1, \dots, x_n] \subseteq K[x_1, \dots, x_n]$ is flat. We prove a stronger result: R is free over $F(R)$ in this case. Since K is free over K^p , $K[x_1^p, \dots, x_n^p]$ is free on the same basis over $K^p[x_1^p, \dots, x_n^p]$. Thus, we need only see that $K[x_1, \dots, x_n]$ is free over $K[x_1^p, \dots, x_n^p]$. It is easy to check that the monomials $x_1^{a_1} \cdots x_n^{a_n}$ such that $0 \leq a_i \leq p-1$ are free basis. \square

We now fill in the missing details of the argument sketched above.

Proposition. *Let $\theta : (R, \mathfrak{m}, K) \rightarrow (S, \mathfrak{n}, L)$ be a homomorphism of local rings that is local, i.e., $\theta(\mathfrak{m}) \subseteq \mathfrak{n}$. Let Q be a finitely generated S -module. Then Q is flat over R if and only if for every injective map $N \hookrightarrow M$ of finite length R -modules, $Q \otimes_R N \rightarrow Q \otimes_R M$ is injective.*

Proof. The condition is obviously necessary. We shall show that it is sufficient. Since tensor commutes with direct limits and every injection $N \hookrightarrow M$ is a direct limit of injections of finitely generated R -modules, it suffices to consider the case where $N \subseteq M$ are finitely generated. Suppose that some $u \in S \otimes_R N$ is such that $u \mapsto 0$ in $S \otimes_R M$. It will suffice to show that there is also such an example in which M and N have finite length. Fix any integer $t > 0$. Then we have an injection

$$N/(m^t M \cap N) \hookrightarrow M/m^t M$$

and there is a commutative diagram

$$\begin{array}{ccc} Q \otimes_R N & \xrightarrow{t} & Q \otimes_R M \\ f \downarrow & & g \downarrow \\ Q \otimes_R (N/(m^t M \cap N)) & \xrightarrow{t'} & Q \otimes_R (M/m^t M) \end{array} .$$