Math 711: Lecture of September 21, 2007

F-finite rings

Let R be a Noetherian ring of prime characteristic $p > 0$. R is called *F-finite* if the Frobenius endomorphism $F : R \to R$ makes R into a module-finite R-algebra. This is equivalent to the assertion that R is module-finite over the subring $F(R) = \{r^p : r \in R\}$, which may also be denoted R^p. When R is reduced, this is equivalent to the condition that $R^{1/p}$ is module-finite over R, since in the reduced case the inclusion $R \subseteq R^{1/p}$ is isomorphic to the homomorphism $F : R \to R$.

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$.*

(a) *R is F-finite if and only if R_{red} is F-finite.*

(b) *R is F-finite if and only and only if $F^e : R \to R$ is module-finite for all e if and only if $F^e : R \to R$ is module-finite for some $e \geq 1$.*

(c) *If R is F-finite, so is every homomorphic image of R.*

(d) *If R is F-finite so is every localization of R.*

(e) *If R is F-finite, so is every algebra finitely generated over R.*

(f) *If R is F-finite, so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

(g) *If (R, m, K) is a complete local ring, R is F-finite if and only if the field K is F-finite.*

(h) *If R is F-finite, so is every ring essentially of finite type over R.*

(i) *If K is a field that is finitely generated as a field over a perfect field, then every ring essentially of finite type over K is F-finite.*

Proof. Parts (c) and (d) both follow from the fact that if B is a finite set of generators for R as $F(R)$-module, the image of B in S will generate S over $F(S)$ if $S = R/J$ and also if $S = W^{-1}R$. In the second case, it should be noted that $F(W^{-1}R)$ may be identified with $W^{-1}F(R)$ because localizing at w and a w^p have the same effect.

For part (a), note that if R is F-finite, so is R_{red} by part (c), since $R_{\text{red}} = R/J$, where J is the ideal of all nilpotent elements. Now suppose that I is any ideal of R such that R/I is F-finite. Let the images of $u_1, \ldots, u_n$ span R/I over the image of $(R/I)^p$, and let $v_1, \ldots, v_h$ generate I over R. Let $A = R^p u_1 + \cdots + R^p u_n$. Then $R = A + Rv_1 + \cdots + Rv_h$. If we substitute the same formula for each copy of R occuring in an Rv_j term on the right, we find thtat

$$R = A + \sum_{i,j} R^p u_i v_j + \sum_{j,j'} Rv'_j v_j.$$

It follows that the $n + nh$ elements u_i and u_iv_j span R/I^2 over the image of $(R/I^2)^p$. Thus, (R/I^2) is F-finite. By a straightforward induction, R/I^{2^k} is F-finite for all k. Hence if $I = J$ is the ideal of nilpotents, we see that R itself is F-finite.

For part (b), note that if $F : R \to R$ is F-finite, so is the e-fold composition. On the other hand, if $F^e : R \to R$ is finite, so is $F^e : S \to S$, where $S = R_{\mathrm{red}}$. Then we have $S \subseteq S^{1/p} \subseteq S^{1/q}$, and since $S^{1/q}$ is a Noetherian S-module, so is $S^{1/p}$. Thus, S is F-finite, and so is R by part (a).

To prove (e), it suffices to consider the case of a polynomial ring in a finite number of variables over R, and, by induction it suffices to consider the case where $S = R[x]$. Likewise, for part (f) we need only show that $R[[x]]$ is F-finite. Let $u_1, \dots, u_n$ span R over R^p. Then, in both cases, the elements u_ix^j, $1 \le i \le n$, $1 \le j \le p-1$, span S over $S^p = R^p[x^p]$ (respectively, $R^p[[x^p]]$).

For (g), note that $K = R/m$, so that if (R, m, K) is F-finite, so is K. If R is complete it is a homomorphic image of a formal power series ring $K[[x_1, \dots, x_n]]$, where K is the residue class field of R. By part (f), if K is F-finite, so is R.

Part (h) is immediate from parts (e) and (d). For part (i) first note that K itself is essentially of finite type over a perfect field, and a perfect field is obviously F-finite. The final statement is then immediate from part (h). □

A proof of the following result of Ernst Kunz would take us far afield. We refer the reader to [E. Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976) 999–1013].

Theorem (Kunz). *Every F-finite ring is excellent.*

We are aiming to prove the following result about F-finite rings:

Theorem (existence of test elements). *Let R be a reduced F-finite ring, and let $c \in R^\circ$ be such that R_c is regular. Then c has a power c^N that is a completely stable big test element.*

This is terrifically useful. Elements $c \in R^\circ$ such that R_c is regular always exist. In any excellent ring,

$$\{P \in \mathrm{Spec}\,(R) : R_P \text{ is regular}\}$$

is open. Since the complement is closed, there is an ideal I such that

$$\mathcal{V}(I) = \{P \in \mathrm{Spec}\,(R) : R_P \text{ is not regular}\}.$$

We refer to this set of primes as the *singular locus* of Spec (R) or of R. Note that if R is reduced, we cannot have $I \subseteq \mathfrak{p}$ for any minimal prime $\mathfrak{p}$ of R, because that would mean the $R_\mathfrak{p}$ is not regular, and $R_\mathfrak{p}$ is a field. Hence, I is not contained in the union of the minimal primes of R, which means that I meets R°. If $c \in I \cap R^\circ$, then R_c is regular: primes that do not contain c cannot contain I. Hence, in a reduced F-finite (or any reduced excellent) ring, there is always an element $r \in R^\circ$ such that R_c is regular, and this means that Theorem above can be applied. Hence:

Corollary. *Every reduced F-finite ring has a completely stable big test element.*

It will take some time before we can prove the Theorem on existence of test elements. Our approach requires studying the notion of a *strongly F-regular* ring. We give the definition below. However, we first want to comment on the notion of an *F-split* ring.

Definition: F-split rings. Let R be a ring of prime characteristic $p > 0$. We shall say that R is *F-split* if, under the map $F : R \to R$, the left hand copy of R is a direct summand of the right hand copy of R.

If R is F-split, $F : R \to R$ must be injective. This is equivalent to the condition that R be reduced. An equivalent condition is therefore that R be reduced and that R be a direct summand of $R^{1/p}$ as an R-module, i.e., there exists an R-linear map $\theta : R^{1/p} \to 1$ such that $\theta(1) = 1$.

Proposition. *Let R be a reduced ring of prime characteristic $p > 0$. The following conditions are equivalent:*

(1) *R is F-split.*

(2) *$R \to R^{1/q}$ splits as a map of R-modules for all q.*

(3) *$R \to R^{1/q}$ splits as a map of R-modules for at least one value of $q > 1$.*

Proof. (1) $\Rightarrow$ (2). Let $\theta : R^{1/p} \to R$ be a splitting. Then for all $q = p^e > 1$, if $q' = p^{e-1}$, we may define a splitting $\theta_e : R^{1/q} \to R^{1/q'}$ by

$$\theta_e(r^{1/q}) = \big(\theta(r^{1/p})\big)^{1/q'}.$$

Thus, the diagram:

$$\begin{array}{ccc} R^{1/q} & \xrightarrow{\theta_e} & R^{1/q'} \\ \simeq\uparrow & & \simeq\uparrow \\ R^{1/p} & \xrightarrow{\theta} & R \end{array}$$

commutes, where the vertical arrows are the isomorphisms $r^{1/p} \mapsto r^{1/q}$ and $r \mapsto r^{1/q'}$, respectively. Of course, $\theta_1 = \theta$. Then θ_e is $R^{1/q'}$-linear and, in particular, R-linear. Hence, the composite map

$$\theta_1 \circ \theta_2 \circ \cdots \circ \theta_e : R^{1/q} \to R$$

gives the required splitting.

(2) $\Rightarrow$ (3) is clear. Finally, assume (3). Then $R \subseteq R^{1/p} \subseteq R^{1/q}$, so that a splitting $R^{1/q} \to R$ may simply be restricted to $R^{1/p}$, and (1) follows. $\square$