

Koszul homology

We define the  $i$ th Koszul homology module  $H_i(x_1, \dots, x_n; M)$  of  $M$  with respect to  $x_1, \dots, x_n$  as the  $i$ th homology module  $H_i(\mathcal{K}_\bullet(x_1, \dots, x_n; M))$  of the Koszul complex.

We note the following properties of Koszul homology.

**Proposition.** *Let  $R$  be a ring and  $\underline{x} = x_1, \dots, x_n \in R$ . Let  $I = (\underline{x})R$ . Let  $M$  be an  $R$ -module.*

- (a)  $H_i(\underline{x}; M) = 0$  if  $i < 0$  or if  $i > n$ .
- (b)  $H_0(\underline{x}; M) \cong M/IM$ .
- (c)  $H_n(\underline{x}; M) = \text{Ann}_M I$ .
- (d)  $\text{Ann}_R M$  kills every  $H_i(x_1, \dots, x_n; M)$ .
- (e) If  $M$  is Noetherian, so is its Koszul homology  $H_i(\underline{x}; M)$ .
- (f) For every  $i$ ,  $H_i(\underline{x}; \_)$  is a covariant functor from  $R$ -modules to  $R$ -modules.
- (g) If

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is a short exact sequence of  $R$ -modules, there is a long exact sequence of Koszul homology

$$\dots \rightarrow H_i(\underline{x}; M') \rightarrow H_i(\underline{x}; M) \rightarrow H_i(\underline{x}; M'') \rightarrow H_{i-1}(\underline{x}; M) \rightarrow \dots$$

- (h) If  $x_1, \dots, x_n$  is a possibly improper regular sequence on  $M$ , then  $H_i(\underline{x}; M) = 0$ ,  $i \geq 1$ .

*Proof.* Part (a) is immediate from the definition. Part (b) follows from the fact that last map in the Koszul complex from  $\mathcal{K}_1(\underline{x}; M) \rightarrow \mathcal{K}_0(\underline{x}; M)$  may be identified with the map  $M^n \rightarrow M$  such that  $(v_1, \dots, v_n) \mapsto x_1 v + \dots + x_n v_n$ . Part (c) follows from the fact that the map  $\mathcal{K}_n(\underline{x}; M) \rightarrow \mathcal{K}_{n-1}(\underline{x}; M)$  may be identified with the map  $M \rightarrow M^n$  such that  $v \mapsto (x_1 v, -x_2 v, \dots, (-1)^{n-1} x_n v)$ .

Parts (d) and (e) are clear, since every term in the Koszul complex is itself a direct sum of copies of  $M$ .

To prove (f), note that if we are given a map  $M \rightarrow M'$ , there is an induced map of complexes

$$\mathcal{K}_\bullet(\underline{x}; R) \otimes M \rightarrow \mathcal{K}_\bullet(\underline{x}; R) \otimes M'.$$

This map induces a map  $H_i(\underline{x}; M) \rightarrow H_i(\underline{x}; M')$ . Checking that this construction gives a functor is straightforward.

For part (g), we note that

$$(*) \quad 0 \rightarrow \mathcal{K}_\bullet(\underline{x}; R) \otimes_R M' \rightarrow \mathcal{K}_\bullet(\underline{x}; R) \otimes_R M \rightarrow \mathcal{K}_\bullet(\underline{x}; R) \otimes_R M'' \rightarrow 0$$

is a short exact sequence of complexes, because each  $\mathcal{K}_j(\underline{x}; R)$  is  $R$ -free, so that the functor  $\mathcal{K}_j(\underline{x}; R) \otimes_R \_$  is exact. The long exact sequence is simply the result of applying the snake lemma to (\*). (This sequence can also be constructed by interpreting Koszul homology as a special case of Tor: we return to this point later.)

Finally, part (h) is immediate by induction from the iterative construction of the Koszul complex as a mapping cone and the Proposition at the top of p. 6 of the Lecture Notes of October 1. The map of augmentations is the map given by multiplication by  $x_n$  from  $M/(x_1, \dots, x_{n-1})M$  to itself, which is injective because  $x_1, \dots, x_n$  is a possibly improper regular sequence.  $\square$

**Corollary.** *Let  $\underline{x} = x_1, \dots, x_n$  be a regular sequence on  $R$  and let  $I = (\underline{x})R$ . Then  $R/I$  has a finite free resolution of length  $n$  over  $R$ , and does not have any projective resolution of length shorter than  $n$ . Moreover, for every  $R$ -module  $M$ ,*

$$\mathrm{Tor}_i^R(R/I, M) \simeq H_i(\underline{x}; M).$$

*Proof.* By part (f) of the preceding Proposition,  $\mathcal{K}_\bullet(\underline{x}; R)$  is acyclic. Since this is a free complex of finitely generated free modules whose augmentation is  $R/I$ , we see that  $R/I$  has the required resolution. Then, by definition of Tor, we may calculate  $\mathrm{Tor}_i^R(R/I, M)$  as  $H_i(\mathcal{K}_\bullet(\underline{x}; R) \otimes_R M)$ , which is precisely  $H_i(\underline{x}; M)$ . To see that there is no shorter projective resolution of  $R/I$ , take  $M = R/I$ . Then

$$\mathrm{Tor}_n(R/I, R/I) = H_n(\underline{x}; R/I) = \mathrm{Ann}_{R/I} I = R/I,$$

by part (c) of the preceding Proposition. If there were a shorter projective resolution, we would have  $\mathrm{Tor}_n(R/I, R/I) = 0$ .  $\square$

### Independence of Koszul homology of the base ring

The following observation is immensely useful. Suppose that we have a ring homomorphism  $R \rightarrow S$  and an  $S$ -module  $M$ . By restriction of scalars,  $M$  is an  $R$ -module. Let  $\underline{x} = x_1, \dots, x_n \in R$  and let  $\underline{y} = y_1, \dots, y_n$  be the images of the  $x_i$  in  $S$ . Note that the actions of  $x_i$  and  $y_i$  on  $M$  are the same for every  $i$ . This means that the complexes  $\mathcal{K}_\bullet(\underline{x}; M)$  and  $\mathcal{K}_\bullet(\underline{y}; M)$  are the same. In consequence,  $H_j(\underline{x}; M) \simeq H_j(\underline{y}; M)$  for all  $j$ , as  $S$ -modules. Note that even if we treat  $M$  as an  $R$ -module initially in calculating  $H_j(\underline{x}; M)$ ,

we can recover the  $S$ -module structure on the Koszul homology from the  $S$ -module structure of  $M$ . For every  $s \in S$ , multiplication by  $s$  is an  $R$ -linear map from  $M$  to  $M$ , and since  $H_i(\underline{x}; \_)$  is a covariant functor, we recover the action of  $s$  on  $H_i(\underline{x}; M)$ .

### Koszul homology and Tor

Let  $R$  be a ring and let  $\underline{x} = x_1, \dots, x_n \in R$ . Let  $M$  be an  $R$ -module. We have already seen that if  $x_1, \dots, x_n$  is a regular sequence in  $R$ , then we may interpret  $H_i(x_1, \dots, x_n; M)$  as a Tor over  $R$ .

In general, we may interpret  $H_i(\underline{x}; M)$  as a Tor over an auxiliary ring. Let  $A$  be any ring such  $R$  is an  $A$ -algebra. We may always take  $A = \mathbb{Z}$  or  $A = R$ . If  $R$  contains a field  $K$ , we may choose  $A = K$ . Let  $\underline{X} = X_1, \dots, X_n$  be indeterminates over  $A$ , and map  $B = A[X_1, \dots, X_n] \rightarrow R$  by sending  $X_j \mapsto x_j$  for all  $j$ . Then  $M$  is also a  $B$ -module, as in the section above, and  $X_1, \dots, X_n$  is a regular sequence in  $B$ .

Hence:

**Proposition.** *With notation as in the preceding paragraph,*

$$H_i(\underline{x}_1, \dots, \underline{x}_n; M) \cong \text{Tor}_i^B(B/(\underline{X})B, M).$$

**Corollary.** *Let  $\underline{x} = x_1, \dots, x_n \in R$ , let  $I = (\underline{x})R$ , and let  $M$  be an  $R$ -module. Then  $I$  kills  $H_i(\underline{x}; M)$  for all  $i$ .*

*Proof.* We use the idea of the discussion preceding the Proposition above, taking  $A = R$ , so that with  $\underline{X} = X_1, \dots, X_n$  we have an  $R$ -algebra map  $B = R[\underline{X}] \rightarrow R$  such that  $X_i \mapsto x_i$ ,  $1 \leq i \leq n$ . Then

$$(*) \quad H_i(\underline{x}; M) \cong \text{Tor}_i^B(B/(\underline{X})B, M).$$

When  $M$  is viewed as a  $B$ -module, every  $X_i - x_i$  kills  $M$ . But  $\underline{X}$  kills  $B/(\underline{X})B$ , and so for every  $i$ , both  $X_i - x_i$  and  $X_i$  kill  $\text{Tor}_i^B(B/(\underline{X})B, M)$ . It follows that every  $x_i = X_i - (X_i - x_i)$  kills it as well, and the result now follows from (\*).  $\square$

### An application to the study of regular local rings

Let  $M$  be a finitely generated  $R$ -module over a local ring  $(R, \mathfrak{m}, K)$ . A *minimal* free resolution of  $M$  may be constructed as follows. Let  $b_0$  be the least number of generators of  $M$ , and begin by mapping  $R^{b_0}$  onto  $M$  using these generators. If

$$R^{b_i} \xrightarrow{\alpha_i} \dots \xrightarrow{\alpha_1} R^{b_0} \xrightarrow{\alpha_0} M \rightarrow 0$$

has already been constructed, let  $b_{i+1}$  be the least number of generators of  $Z_i = \text{Ker}(\alpha_i)$ , and construct  $\alpha_{i+1} : R^{b_{i+1}} \rightarrow R^{b_i}$  by mapping the free generators of  $R^{b_{i+1}}$  to a minimal set of generators of  $Z_i \subseteq R^{b_i}$ . Think of the linear maps  $\alpha_i$ ,  $i \geq 1$ , as given by matrices. Then it is easy to see that a free resolution for  $M$  is minimal if and only if all of the matrices  $\alpha_i$  for  $i \geq 1$  have entries in  $\mathfrak{m}$ . We have the following consequence: