Chapter 1: Section 1.6 **Inverse Functions and Logarithms**

Definition: A function $f(x)$ is called a one-to-one if it never takes on the same value twice; that is

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

Example 1:

$f(x) = x^2$ — Not one-to-one

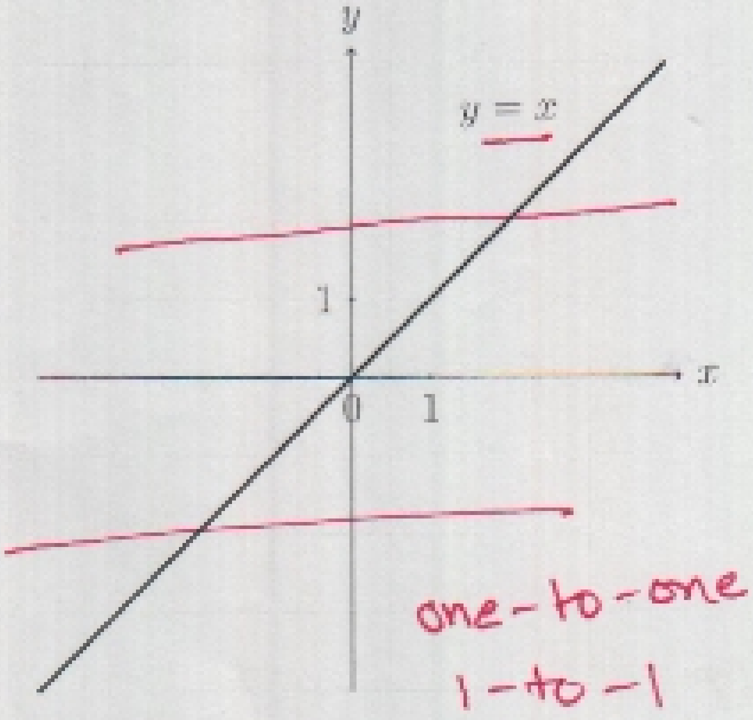
$x = 1 \quad f(1) = 1^2 = 1$

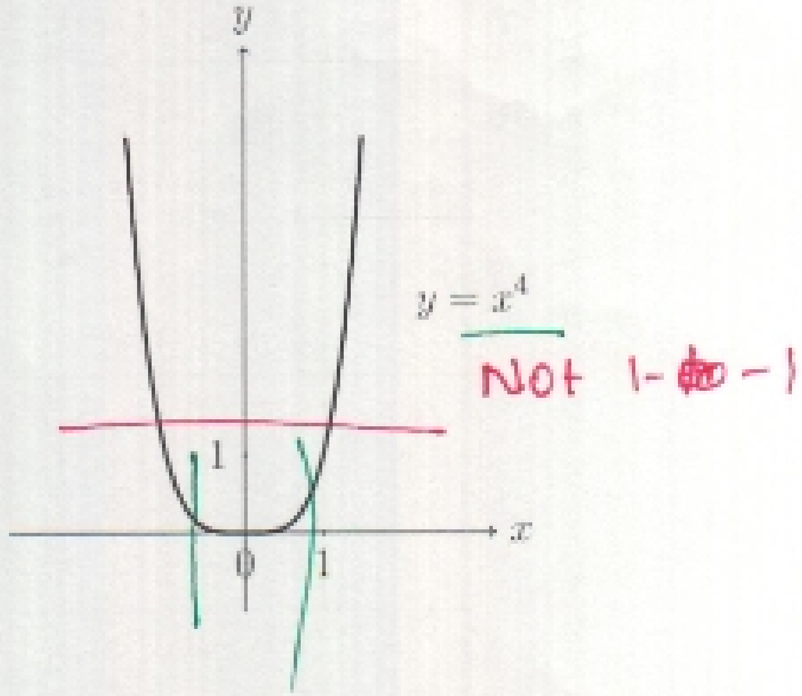
$x = -1 \quad f(-1) = (-1)^2 = 1$

$f(x) = x^2, \; x \geq 0$

Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 2: Which of the following functions are one-to-one?





Definition: If a function f is a one-to-one function with domain A and range B, then its inverse, denoted as f^{-1}, has a domain B and range A and is defined as

$$f(x) = y \iff x = f^{-1}(y)$$

Domain f = Range f^{-1}

Range f = Domain f^{-1}

for any y in B. Thus, if (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}. Thus, if f is one-to-one, then the graph of its inverse, f^{-1}, is the reflection about the line $y = x$.

Example 3: If f is a one-to-one function and given $f(1) = 3$ and $f(3) = 1$, then find

(a) $f^{-1}(3)$ $\quad f(1) = 3$, $1 = f^{-1}(3)$ $\quad f(3) = 1$, $3 = f^{-1}(1)$

$f^{-1}\circ f$ (a) $f^{-1}(f(3))$ $\quad f^{-1}(f(3)) = f^{-1}(1) = 3$ $\quad f^{-1}(f(a)) = f(f^{-1}(a)) = a$

Composition

How to Find the Inverse Function of a One-to-One Function, y = f(x):

(1) In the given function $y = f(x)$, interchange x and y.

(2) Solve the resulting equation for y to get the inverse function of $f(x)$.

> **CAUTION:** $f^{-1}(x)$ is not the same as $\frac{1}{f(x)}$, i.e. $f^{-1}(x) \neq \frac{1}{f(x)}$

$f^{-1}(x) \neq \frac{1}{f(x)}$ $\quad x^{-2} = \frac{1}{x^2}$

Example 4: Find the inverse of the function $f(x) = 5x - 1$ and also state the domain and range of the function f and its inverse f^{-1}.

$f(x) = 5x+1$

Domain: $\mathbb{R}$ - Real numbers

Range: $\mathbb{R}$

$y = f(x)$

$y = 5x+1$

(i) $x = 5y+1$

$5y+1 = x$

$-1 \quad -1$

$5y = x-1$

$\div 5 \quad y = \frac{x-1}{5}$

$f^{-1}(x) = \frac{x-1}{5}$

D: $\mathbb{R}$

R: $\mathbb{R}$

Definition: The inverse of an exponential function, $y = b^x$, is called a Logarithmic, provided $b > 0$ and $b \neq 1$.

Properties of Logarithmic Function: If b, M, and N are positive real numbers, $b \neq 1$, and x is a real number.

$\log_{10} 1 = 0 \quad \log_5 1 = 0$

$\log_{10} 10 = 1 \quad \log_5 5 = 1$

- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b MN = \log_b M + \log_b N$ — $\log_b(MNP) = \log_b M + \log_b N + \log_b P$
- $\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$
- $\log_b M^N = N \log_b M$ $\quad \log_3 x^2 = 2\log_3 x$
- $\log_b M = \log_b N$ if and only if $M = N$

$\log_2(x+3) = \log_2 5 \Rightarrow x+3 = 5$

Example 5: Express the given quantity as a single logarithm:

$$\log_{10}(1+x^5) + \frac{1}{2}\log_{10}(x) - \log_{10}(\cos x)$$

$= \log_{10}(1+x^5) + \log_{10}(x)^{1/2} - \log_{10}(\cos x)$

$= \log_{10}((1+x^5)\cdot x^{1/2}) - \log_{10}(\cos x) = \log_{10}\left(\frac{\sqrt{x}(1+x^5)}{\cos x}\right)$

Example 6: Find the exact value of the expression:

$$\log_3 12 - \log_3 28 + \log_3 63.$$

$= \log_3\left(\frac{12}{28}\right) + \log_3 63$

$= \log_3\left(\frac{3}{7}\right) + \log_3(63) = \log_3\left(\frac{3}{7} * 63\right)$

$= \log_3(27)$

$= \log_3(3^3)$

$= 3\log_3 3 = 3 * 1$

$= 3$

$27 = 3^3$

Example 7: Solve the following for x:

(1) $\log_{10}(x+2) + \log_{10}(x-2) = \log_{10} 12$

$\log_{10}((x+2)(x-2)) = \log_{10} 12$

$\log_{10}(x^2-4) = \log_{10} 12$

$x^2 - 4 = 12$

$+4 \quad +4$

$x^2 = 16$

$x = \pm\sqrt{16}$

$x = \pm 4$

$x = 4$, ~~$x = -4$~~

(2) $\log_{10} x = \frac{2}{3}\log_{10} 27 - 2\log_{10} 2 - \log_{10} 3$

$= \log_{10} 27^{2/3} - \log_{10} 2^2 - \log_{10} 3$

$= \log(3^3)^{2/3} - \log_{10} 4 - \log_{10} 3$

$\log_{10} x = \log_{10} 9 - \log_{10} 4 - \log_{10} 3$

$\log_{10} x = \log_{10}\left(\frac{9}{4}\right) - \log_{10} 3$

$= \log_{10}\left(\frac{9}{4} \div 3\right) = \log_{10}\left(\frac{9}{4} * \frac{1}{3}\right)$

Definition: The logarithm with base e is called natural logarithm and has a special notation:

$e = 2.71$

$$\log_e x = \ln x$$

$\ln x \quad \log_3 x$

Example 8: Using the natural logarithm, find the value of x in the following:

(1) $\ln(1-2x) = 5$

$\log_e(1-2x) = 5$

$1 - 2x = e^5$

$-1 \quad -1$

$-2x = e^5 - 1$

$\div -2$

$x = \frac{-1}{2}(e^5 - 1)$

$\ln x$

$\log x$

$\log_5 x$

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